

Lab 3. Continuous Random Variables

Objectives

- Continuous distributions in R: exponential, Γ , and χ^2
- Related statistics, properties, and simulation

Uniform random variable – Continuous Case

$$X \sim \text{uniform}(a, b)$$

Then,

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

$$\mu = \frac{a+b}{2}, \quad \sigma^2 = \frac{(b-a)^2}{12}$$

$$\mu = E(X) = \int_a^b x \cdot f(x) dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{a+b}{2}$$

$$E(X^2) = \int_a^b x^2 \cdot f(x) dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3} = \frac{a^2 + ab + b^2}{3}$$

$$\sigma^2 = E(X^2) - \{E(X)\}^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

Statistics programs including R generate uniform random numbers between 0 and 1, by default. You can ask for uniform random numbers between 5 and 10, for example. Here is how to simulate continuous uniform random numbers.

```
x <- runif(100)      #uniform random numbers between 0 and 1
summary(x)
x <- runif(100,5,10) #uniform random numbers between 5 and 10
summary(x)
plot(x)
mean(x); var(x); sd(x)
((10-5)^2)/12
sqrt(((10-5)^2)/12)
```

The last part is to check if the sample statistics are close to the theoretical ones. For a continuous uniform random variable, $\mu = \frac{a+b}{2}$ and $\sigma^2 = \frac{(b-a)^2}{12}$.

Ex 1: Uniform

1. Let $U_1 \sim \text{uniform}(0, 1)$ and another independent random variable $U_2 \sim \text{uniform}(0, 1)$. Then,

$$X_1 = \sqrt{(-2 \log U_1)} \cdot \cos(2\pi U_2)$$

$$X_2 = \sqrt{(-2 \log U_1)} \cdot \sin(2\pi U_2)$$

are two independent random variables and both will be normally distributed with $\mu = 0$ and $\sigma = 1$. This is a well-known transformation to generate standard normal random variates from the uniform random variables and is known as the “**Box-Müller Transformation**”. We can explore this using uniform $(0, 1)$ random numbers.

```
u1 <- runif(1000)
u2 <- runif(1000)
x1 <- (sqrt(-2*log(u1)))*(cos(2*pi*u2))
x2 <- (sqrt(-2*log(u1)))*(sin(2*pi*u2))
plot(x1,x2)
plot(density(x1))
plot(density(x2))
shapiro.test(x1)
shapiro.test(x2)
qqnorm(x1);qqline(x1)
qqnorm(x2);qqline(x2)
```

2. Let $U \sim \text{uniform}(0, 1)$. Then,

$$F^{-1}(U) \sim X$$

What this means is that for any continuous random variable with a cdf F , we can generate the random variables by plugging the uniform random numbers into the *inverse* of the cdf function. Then, the end result is, voilà, the random variable with the desired distribution. This is another well-known transformation using the uniform random variables and is known as the “**Probability Integral Transformation**”. Let’s explore this using uniform $(0, 1)$ random numbers.

```
u <- runif(1000)
summary(u)
x1 <- qnorm(u)           #to generate standard normal random variables
shapiro.test(x1)
qqnorm(x1); qqline(x1)
x2 <- qnorm(u,100,13)    #to generate normal random variables mean 100, sd 13
shapiro.test(x2)
qqnorm(x2); qqline(x2)
x3 <- qchisq(u,20)       #to generate random variables having the Chi-square dist with df 20
plot(density(x3))
qqnorm(x3);qqline(x3)
mean(x3); var(x3)       #Chi-square dist with df 20 will have mean 20, variance 40
```

3. Ace Heating and Air Conditioning Service finds that the amount of time a repairman needs to fix a furnace is uniformly distributed between 1.5 and four hours. Let X = the time needed to fix a furnace. Then $X \sim U(1.5, 4)$.
- Find the probability that a randomly selected furnace repair requires more than two hours.
 - Find the probability that a randomly selected furnace repair requires less than three hours.
 - Find the 30th percentile of furnace repair times.
 - The longest 25% of furnace repair times take at least how long? (In other words, find the minimum time for the longest 25% of the repair times.) What percentile does this represent?
 - Find the mean and standard deviation of X .

Let $X \sim \text{uniform}(1.5, 4)$. Then $f(x) = \frac{1}{4-1.5} = \frac{1}{2.5} = 0.4$, $x \in (1.5, 4)$.

$$P(X > 2) = \int_2^4 f(x) dx = 0.4 \int_2^4 dx = 0.8$$

$$P(X < 3) = \int_{1.5}^3 f(x) dx = 0.4 \int_{1.5}^3 dx = 0.6$$

$$P(X < k) = \int_{1.5}^k f(x) dx = 0.4 \int_{1.5}^k dx = 0.3 \Rightarrow k = 2.25$$

$$P(X > k) = \int_k^4 f(x) dx = 0.4 \int_k^4 dx = 0.25 \Rightarrow k = 3.375$$

The longest 25% of furnace repairs take at least 3.375 hours (3.375 hours or longer). **Note:** Since 25% of repair times are 3.375 hours or longer, that means that 75% of repair times are 3.375 hours or less. 3.375 hours is the 75th percentile of furnace repair times

$$\mu = \frac{a+b}{2} = \frac{1.5+4}{2} = 2.75, \quad \sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(4-1.5)^2}{12}} = 0.7217 \text{ (hours)}$$

Exponential random variable

$$X \sim \text{Exp}(\theta)$$

Then,

$$\text{pdf } f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad 0 \leq x$$

$$\text{cdf } F(x) = P(X \leq x) = \int_0^x f(t) dt = 1 - e^{-\frac{x}{\theta}}, \quad 0 \leq x$$

$$\mu = \theta, \quad \sigma^2 = \theta^2$$

$$\mu = E(X) = \int_0^\infty x \cdot f(x) dx = \frac{1}{\theta} \int_0^\infty x e^{-\frac{x}{\theta}} dx = \frac{1}{\theta} \cdot \theta^2 = \theta$$

$$E(X^2) = \int_0^\infty x^2 \cdot f(x) dx = \frac{1}{\theta} \int_0^\infty x^2 e^{-\frac{x}{\theta}} dx = 2\theta^2$$

$$\sigma^2 = E(X^2) - \{E(X)\}^2 = \theta^2$$

The **exponential distribution** is often concerned with the amount of time until some specific event occurs. For example, the amount of time (beginning now) until an earthquake occurs has an exponential distribution. Other examples include the length, in minutes, of long distance business telephone calls, and the amount of time, in months, a car battery lasts. It can be shown, too, that the value of the change that you have in your pocket or purse approximately follows an exponential distribution. For an exponential random variable, there are fewer large values and more small values. For example, the amount of money customers spend in one trip to the supermarket follows an exponential distribution. There are more people who spend small amounts of money and fewer people who spend large amounts of money. The exponential distribution is widely used in the field

of natural science and reliability. Reliability deals with the amount of time a product lasts.

For example, let X = amount of time (in minutes) a postal clerk spends with his or her customer. The time is known to have an exponential distribution with the average amount of time equal to four minutes.

$$X \sim \text{Exp}(4).$$

We can calculate the probability that a clerk spends four to five minutes with a randomly selected customer as the following.

$$P(4 \leq X \leq 5) = \int_4^5 f(x) dx = \frac{1}{4} \int_4^5 e^{-\frac{x}{4}} dx = -e^{-\frac{x}{4}} \Big|_{x=4}^{x=5} = e^{-1} - e^{-5/4} = 0.08137$$

Can you tell how long it will take to finish half of all customers? (Can you find the 50th percentile?) This is a question that can be formulated as solving for k from the following setup:

$$P(X \leq k) = F(k) = 1 - e^{-0.25k} = 0.5$$

$$\therefore k = \frac{-\ln 0.5}{0.25} = 2.772589$$

Half of all customers are finished within 2.8 minutes. All of these questions can also be answered by R.

```
pexp(5,0.25)-pexp(4,0.25)
qexp(0.5,0.25)
```

Here is how to simulate 1000 exponential random numbers with $\theta = 4$.

```
x <- rexp(1000,0.25)
plot(density(x))
mean(x); var(x); sd(x)
```

The last part is to check if the sample mean and sample sd are close to the theoretical mean and sd. For an exponential random variable, μ and σ are all equal to θ .

Ex 2: Exponential

1. On the average, a certain computer part lasts ten years. The length of time the computer part lasts is exponentially distributed.
 - (a) What is the probability that a computer part lasts more than 7 years?
 - (b) On the average, how long would five computer parts last if they are used one after another?
 - (c) Eighty percent of computer parts last *at most* how long?
 - (d) What is the probability that a computer part lasts between nine and 11 years?

Let $X \sim \text{Exp}$ with $\theta = 0.1$.

```
1-pexp(7,0.1)
5*10
qexp(0.8,0.1)
pexp(11,0.1)-pexp(9,0.1)
```

2. The time spent waiting between events is often modeled using the exponential distribution. For example, suppose that an average of 30 customers per hour arrive at a store and the time between arrivals is exponentially distributed.
 - (a) On average, how many minutes elapse between two successive arrivals?
 - (b) When the store first opens, how long on average does it take for three customers to arrive?
 - (c) After a customer arrives, find the probability that it takes less than one minute for the next customer to arrive.
 - (d) After a customer arrives, find the probability that it takes more than five minutes for the next customer to arrive.
 - (e) Seventy percent of the customers arrive within how many minutes of the previous customer?
 - (f) Is an exponential distribution reasonable for this situation?

Since we expect 30 customers to arrive per hour, we expect on average one customer to arrive every two minutes on average. This also means that it will take six minutes on average for three customers to arrive. Let X = the time between arrivals (in minutes). Then, $X \sim \text{Exp}(2)$.

```
pexp(1,0.5)
1-pexp(5,0.5)
qexp(0.7,0.5)
```

From the last calculation, we find that seventy percent of customers arrive within 2.41 minutes of the previous customer. Regarding the last question, this model assumes that a single customer arrives at a time, which may not be reasonable since people often shop in groups, leading to several customers arriving at the same time. It also assumes that the flow of customers does not change throughout the day, which is not valid if some times of the day are busier than others.

3. According to the American Red Cross, about one out of nine people in the U.S. have Type B blood. Suppose the blood types of people arriving at a blood drive are independent. In this case, the number of Type B blood types that arrive roughly follows the Poisson distribution.
 - (a) If 100 people arrive, how many on average would be expected to have Type B blood?
 - (b) What is the probability that over 10 people out of these 100 have type B blood?
 - (c) What is the probability that more than 20 people arrive before a person with type B blood is found?
- (a) $100/9 = 11.11$
 - (b) $P(X > 10) = 1 - P(X \leq 10)$. Use R to find `1-ppois(10,100/9)`.
 - (c) The number of people with Type B blood encountered roughly follows the Poisson distribution, so the number of people X who arrive between successive Type B arrivals is roughly exponential with mean $\theta = 9$. $P(X > 20) = 1 - P(X \leq 20)$. Use R to find

$1 - \text{pexp}(20, 1/9)$ or $1 - \text{pgamma}(20, 1, 1/9)$.

4. Let $X \sim \text{Exp}(\theta = 2)$. Construct the probability distribution graph of X .

```
x <- 0:10
prob <- dexp(x, 0.5)
cdf <- pexp(x, 0.5)
plot(x, prob, type="o", col="red", ylim=c(0, 1))
text(0:10, prob, labels=round(prob, 4), pos=1, cex=0.6, offset=0.3)
lines(0:10, cdf, pch=16, type="o", col="blue")
text(0:10, cdf, labels=round(cdf, 4), pos=3, cex=0.6, offset=0.3)
legend(locator(1), c("pdf", "cdf"), pch=c(1, 16), col=c("red", "blue"))
```

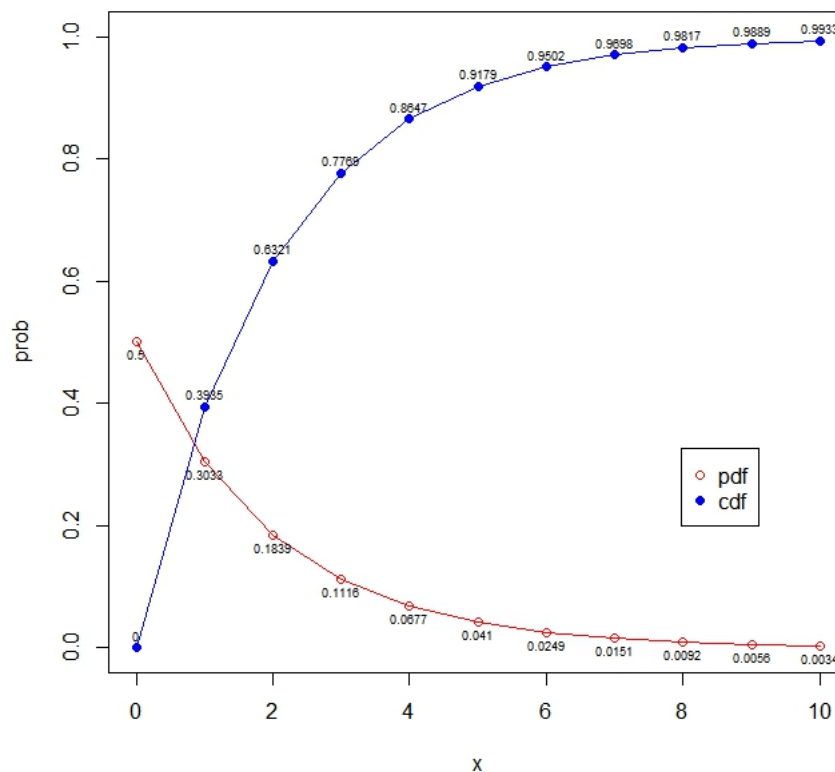


Figure 1: pdf and cdf of an exponential random variable with $\theta = 2$

The following R codes compare three exponential distributions with $\theta = 1, 2$, & 5 .

```
x <- seq(0, 20, 0.01)
pdf0.5 <- dexp(x, 0.5)
pdf0.2 <- dexp(x, 0.2)
pdf1 <- dexp(x, 1)
plot(x, pdf1, ylim=c(0, 1), type="l", lty=1)
lines(x, pdf0.2, ylim=c(0, 1), type="l", lty=2, col="2")
lines(x, pdf0.5, ylim=c(0, 1), type="l", lty=3, col="4")
```

```

text(locator(1),expression(theta=="1"))
text(locator(1),expression(theta=="5"))
text(locator(1),expression(theta=="2"))

```

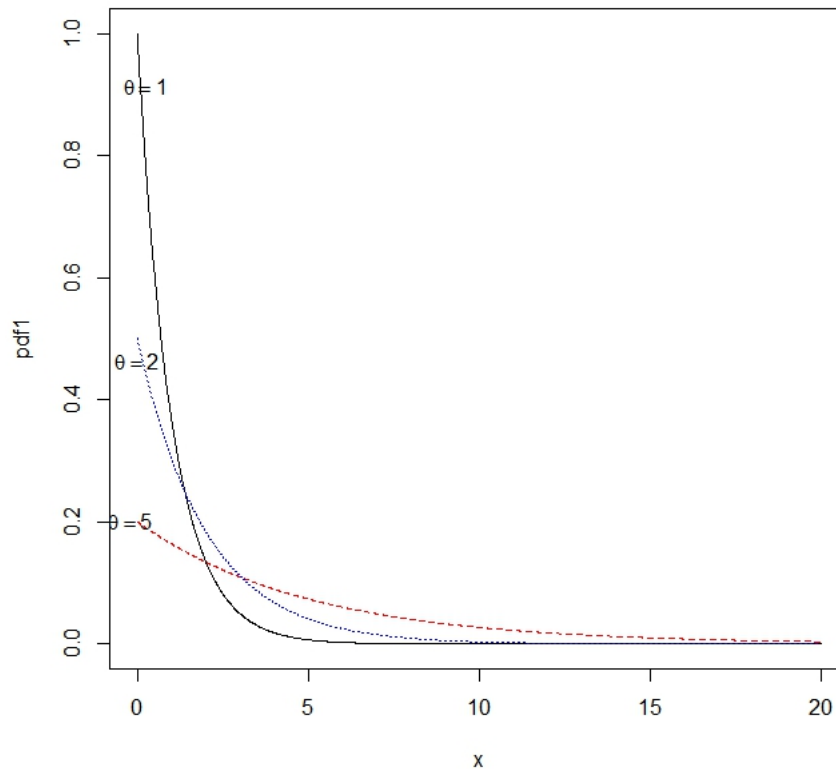


Figure 2: pdf of exponential random variables with $\theta = 1, 2, 5$

Ex 3: Memorylessness of the Exponential Distribution

Suppose that the amount of time between customers is exponentially distributed with a mean of two minutes ($X \sim \text{Exp}(0.5)$). Suppose also that five minutes have elapsed since the last customer arrived. Since an unusually long amount of time has now elapsed, it would seem to be more likely for a customer to arrive within the next minute. With the exponential distribution, this is not the case – the additional time spent waiting for the next customer does not depend on how much time has already elapsed since the last customer. This is referred to as the *memoryless* property. Specifically, the memoryless property says that

$$P(X > r + t \mid X > r) = P(X > t) \quad \forall r \geq 0 \text{ and } t \geq 0$$

For example, if five minutes have elapsed since the last customer arrived, then the probability that more than one minute will elapse before the next customer arrives is computed by using $r = 5$ and

$t = 1$ in the foregoing equation.

$$P(X > 5 + 1 | X > 5) = P(X > 1) = e^{(-0.5)(1)} \approx 0.6065.$$

This is the same probability as that of waiting more than one minute for a customer to arrive after the previous arrival. The exponential distribution is often used to model the longevity of an electrical or mechanical device. For example, the lifetime of a certain computer part has the exponential distribution with a mean of ten years ($X \sim \text{Exp}(0.1)$). The memoryless property says that knowledge of what has occurred in the past has no effect on future probabilities. It means that an old part is not any more likely to break down at any particular time than a brand new part. In other words, the part stays as good as new until it suddenly breaks. For example, if the part has already lasted ten years, then the probability that it lasts another seven years is $P(X > 17 | X > 10) = P(X > 7) = 0.4966$.

Ex 4: Relationship between the Poisson and the Exponential Distribution

There is an interesting relationship between the exponential distribution and the Poisson distribution. Suppose that the time that elapses between two successive events follows the exponential distribution with a mean of θ units of time. Also assume that these times are independent, meaning that the time between events is not affected by the times between previous events. If these assumptions hold, then the number of events per unit time follows a Poisson distribution with mean $\lambda = \frac{1}{\theta}$. Recall from the chapter on Discrete Random Variables that if X has the Poisson distribution with mean λ , then $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$. Conversely, if the number of events per unit time follows a Poisson distribution, then the amount of time between events follows the exponential distribution.

In the (approximate) Poisson process with mean λ , the waiting time until the occurrence has an exponential distribution. Let W = the waiting time until the α th occurrence and let's find the distribution of W .

The cdf of $W(w \geq 0)$ is given by

$$\begin{aligned} F(w) &= P(W \leq w) = 1 - P(W > w) \\ &= 1 - P(\text{fewer than } \alpha \text{ occurrences in } [0, w]) \\ &= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \end{aligned}$$

The pdf of $W(w \geq 0)$ is given by

$$\begin{aligned}
 f(w) &= F'(w) = \lambda e^{-\lambda w} - e^{-\lambda w} \sum_{k=1}^{\alpha-1} \left\{ \frac{k(\lambda w)^{k-1} \lambda}{k!} - \frac{(\lambda w)^k \lambda}{k!} \right\} \\
 &= \lambda e^{-\lambda w} - e^{-\lambda w} \left\{ \frac{1(\lambda w)^0 \lambda}{1!} - \frac{(\lambda w)^1 \lambda}{1!} \right\} + \left\{ \frac{2(\lambda w)^1 \lambda}{2!} - \frac{(\lambda w)^2 \lambda}{2!} \right\} + \left\{ \frac{3(\lambda w)^2 \lambda}{3!} - \frac{(\lambda w)^3 \lambda}{3!} \right\} \\
 &\quad + \dots + \left\{ \frac{(\alpha-1)(\lambda w)^{\alpha-2} \lambda}{(\alpha-1)!} - \frac{(\lambda w)^{\alpha-1} \lambda}{(\alpha-1)!} \right\} \\
 &= \lambda e^{-\lambda w} - e^{-\lambda w} \left\{ \lambda - \frac{(\lambda w)^{\alpha-1} \lambda}{(\alpha-1)!} \right\} \\
 &= \frac{(\lambda w)^{\alpha-1} \lambda}{(\alpha-1)!} e^{-\lambda w} \sim \Gamma\left(\alpha, \frac{1}{\lambda}\right)
 \end{aligned}$$

For example, let X = the number of α particle emissions of ^{14}C (carbon-14) that are counted by a Geiger counter each second. Assume that the distribution of X is Poisson with mean 16. Let W = the time in seconds before the seventh count is made. Then, $W \sim \Gamma\left(7, \frac{1}{16}\right)$ and we can find $P(W \leq 0.5)$ by

```
pgamma(0.5,7,16)
[1] 0.6866257
```

Here is how the pdf of $\Gamma(7, 16)$ would look like.

```
x <- seq(0,1.3,0.01)
pdf_x <- dgamma(a,7,16)
plot(x,pdf_x,type="l",col=4)
#To check the sample mean, variance and sd
x <- rgamma(1000,7,16)
mean(x);var(x);sd(x)
7/16; 7/(16^2); sqrt(7/(16^2))
```

Check if the sample statistics (\bar{X}, s^2, s) are the same as the (theoretical) population parameters (μ, σ^2, σ) , respectively.

Gamma, Exponential and χ^2 random variable

$$X \sim \Gamma(\alpha, \theta),$$

Then,

$$\begin{aligned}
 \text{pdf } f(x) &= \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad 0 \leq x \\
 \mu &= \alpha\theta, \quad \sigma^2 = \alpha\theta^2
 \end{aligned}$$

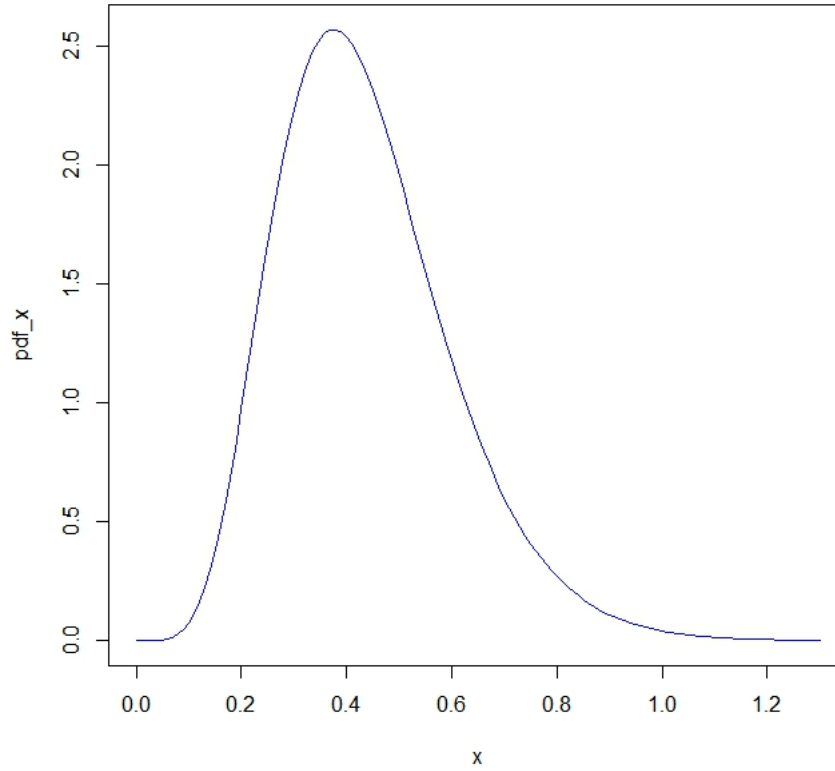


Figure 3: pdf of Γ random variable with $\alpha = 7$ and $\theta = 1/16$

$$\mu = E(X) = \int_0^{\infty} x \cdot f(x) dx = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} x^{\alpha} e^{-x/\theta} dx$$

Let $\frac{x}{\theta} = t$, then

$$E(X) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \theta^{\alpha+1} \int_0^{\infty} t^{\alpha} e^{-t} dt = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \theta^{\alpha+1} \Gamma(\alpha+1) = \alpha\theta$$

The last part is by the definition of gamma function and the fact that $\Gamma(\alpha) = (\alpha-1)!$ and

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt = (\alpha-1)!$$

In a similar way,

$$E(X^2) = \int_0^{\infty} x^2 \cdot f(x) dx = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} x^{\alpha+1} e^{-x/\theta} dx = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \theta^{\alpha+2} \int_0^{\infty} t^{\alpha+1} e^{-t} dt = \frac{\theta^{\alpha+2} \Gamma(\alpha+2)}{\Gamma(\alpha)\theta^{\alpha}} = (\alpha+1)\alpha\theta^2$$

$$\sigma^2 = E(X^2) - \{E(X)\}^2 = \alpha\theta^2$$

Now is the time to learn the relationship among these three distributions: Γ , exponential, and χ^2 .

Both the exponential distribution and the χ^2 distribution are special case of the Γ distribution and they are related in the following way:

Distribution	pdf	mean & sd
$X \sim \Gamma(\alpha, \theta)$	$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad 0 \leq x$	$\mu = \alpha\theta, \quad \sigma^2 = \alpha\theta^2$
$X \sim \text{Exp}(\lambda)$	$f(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}, \quad 0 \leq x$	$\mu = \frac{1}{\lambda}, \quad \sigma^2 = \frac{1}{\lambda^2}$
$X \sim \chi^2(r)$	$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \quad 0 \leq x$	$\mu = r, \quad \sigma^2 = 2r$

That is,

- $X \sim \text{Exp}(\lambda)$ is a special case of the Γ with $\alpha = 1$ and $\theta = \frac{1}{\lambda}$.
- $X \sim \chi^2(r)$ is a special case of the Γ with $\alpha = \frac{r}{2}$ and $\theta = 2$.

Here is how to plot the pdf of Γ distributions with a variety of α values, and $\theta = 2$.

```
x <- seq(0,30,0.05)
pdf_gam_1_2 <- dgamma(x,1,1/2)
pdf_gam_2_2 <- dgamma(x,2,1/2)
pdf_gam_3_2 <- dgamma(x,3,1/2)
pdf_gam_4_2 <- dgamma(x,4,1/2)
pdf_gam_5_2 <- dgamma(x,5,1/2)
pdf_gam_10_2 <- dgamma(x,10,1/2)
plot(x,pdf_gam_1_2,type="l",ylab="")
lines(x,pdf_gam_2_2,type="l",lty=2,col=2)
lines(x,pdf_gam_3_2,type="l",lty=3,col=3)
lines(x,pdf_gam_4_2,type="l",lty=4,col=4)
lines(x,pdf_gam_5_2,type="l",lty=5,col=5)
lines(x,pdf_gam_10_2,type="l",lty=6,col=6)
text(locator(1),expression(theta=="2"))
text(locator(1),expression(alpha=="1"))
text(locator(1),expression(alpha=="2"))
text(locator(1),expression(alpha=="3"),cex=0.6)
text(locator(1),expression(alpha=="4"),cex=0.6)
text(locator(1),expression(alpha=="5"),cex=0.6)
text(locator(1),expression(alpha=="10"))
```

Now, let's simulate 1000 Γ random numbers with $\alpha=10$, $\theta=2$.

```
x <- rgamma(1000,10,0.5)
summary(x)
plot(density(x))
mean(x); sd(x); var(x)
10*2; sqrt(10*2*2)
```

The last part is to check if the sample mean and sample sd are close to the theoretical mean and sd. For the Γ random variable, $\mu = \alpha\theta$ and $\sigma^2 = \alpha\theta^2$. Some more examples of a Γ random variable $X \sim (\alpha, \theta)$ are shown below.

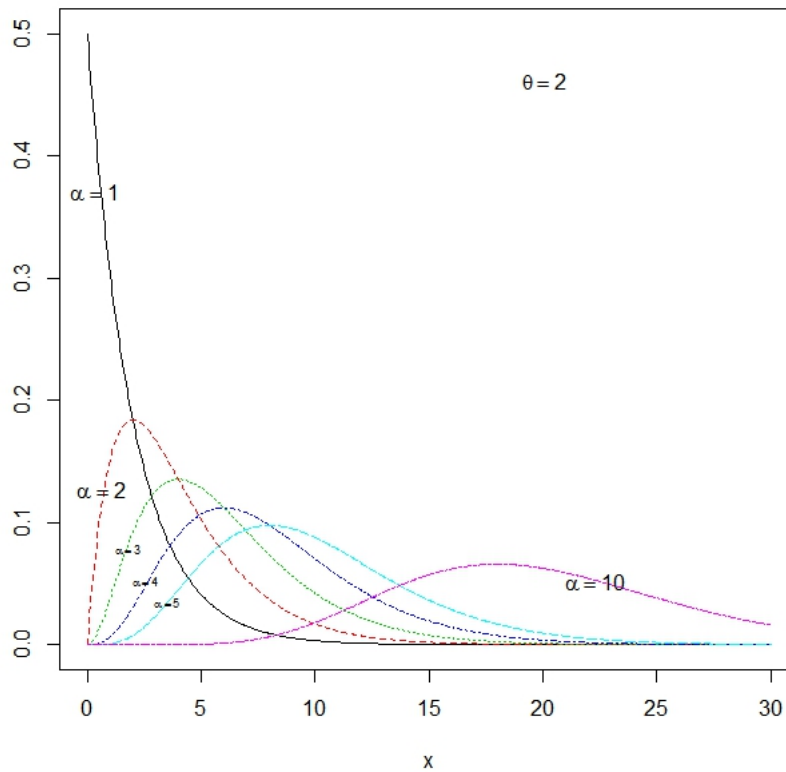


Figure 4: pdf of Γ random variables with a variety of α values and $\theta = 2$

Ex 5: Γ

1. Let $X \sim \Gamma$ with $\alpha = 2$ and $\theta = 4$. Find $P(X < 5)$.

`pgamma(2,0.25)`

2. Let $X \sim \chi^2(17)$, find

- (a) $P(X > 27.59)$.
- (b) $P(6.408 < X < 27.59)$.
- (c) $\chi^2_{0.95}(17)$.
- (d) $\chi^2_{0.25}(17)$.

3. If 15 observations are taken independently from a χ^2 distribution with 4 degrees of freedom, find the probability that at most 3 of the 15 observations exceed 7.779.

Selected Problems

1. At a police station in a large city, calls come in at an average rate of four calls per minute. Assume that the time that elapses from one call to the next has the exponential distribution. Take note that we are concerned only with the rate at which calls come in, and we are ignoring the time spent on the phone. We must also assume that the times spent between calls are independent. This means that a particularly long delay between two calls does not mean that there will be a shorter waiting period for the next call. We may then deduce that the total number of calls received during a time period has the Poisson distribution.
 - (a) Find the average time between two successive calls.
 - (b) Find the probability that after a call is received, the next call occurs in less than ten seconds.
 - (c) Find the probability that exactly five calls occur within a minute.
 - (d) Find the probability that less than five calls occur within a minute.
 - (e) Find the probability that more than 40 calls occur in an eight-minute period.
2. Cars arrive at a tollbooth at a mean rate of 5 cars every 10 minutes according to a Poisson process. Find the probability that the toll collector will have to wait longer than 26.30 minutes before collecting the eighth toll.
3. An aluminum screen 2 feet in width has, on average, three flaws in a 100-foot roll.
 - (a) Define an appropriate random variable X and state the distribution and its relevant parameters.
 - (b) Find the probability that the first 40 feet in a roll contains no flaws.
 - (c) What assumptions did you make to solve the part (b)?
 - (d) Plot a graph of the pdf of X .
4. A bakery sells rolls in units of a dozen. The demand X (in 1,000 units) for rolls has a gamma distribution with parameters $\alpha = 3$, $\theta = 0.5$, where θ is in units of days per 1,000 units of rolls. It costs \$2 to make a unit that sells for \$5 on the first day when the rolls are fresh. Any leftover units are sold on the second day for \$1. How many units should be made to maximize the expected value of the profit?