Chapter 5. Distribution of Functions of Random Variables

Three Basic Methods

- cdf method
- pdf method (i.e., transformation-of-variables method)
- mgf method

CDF Method

Usually applicable when $Y = T(X)$ is 1-dimensional continuous random variable. Also, this method works well to find the distribution of $\max(X_1, \dots, X_n)$ or $\min(X_1, \dots, X_n)$.

1. Rewrite Y in terms of X: $(Y \le y) = \{T(X) \le y\} = (X \in \mathscr{A}_y)$
2. Find the cdf of Y using the distribution of X: $F_Y(y) = P(Y \le y) = P(X \in \mathscr{A}_y)$
3. Find the pdf of Y from the cdf: $f_y(y) = \frac{\partial}{\partial y} F_Y(y)$

Ex 1. $X \sim uniform\,(0, 1)$. Find the distribution of $Y = -\ln(1 - X)$.

Answer: First, note that $y > 0$.

$$F(y) = P(Y \le y) = P\{-\ln(1-X) \le y\} = P\left(X \le 1 - e^{-y}\right) = \begin{cases} 1 - e^{-y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f(y) = \frac{\partial}{\partial y} F(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

i.e., $Y \sim$ exponential with $(\lambda = 1)$.

Ex 2. Let $X_1, X_2, \dots, X_n$ be iid continuous random variables with pdf $f(\cdot)$. Find
(a) distribution of $Y = \max(X_1, \dots, X_n)$
(b) distribution of $Z = \min(X_1, \dots, X_n)$

Answer:

(a) $F_Y(y) = P\{\max(X_1, \dots, X_n) \le y\} = P(X_1 \le y, \dots, X_n \le y) = \prod_{i=n}^{n} P(X_i \le y) = \{F_X(y)\}^n$

$$f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = n\{F_X(y)\}^{n-1} f_X(y)$$

(b) $F_Z(z) = P\{\min(X_1, \ldots, X_n) \leq z\} = 1 - P(X_1 > z, \ldots, X_n > z) = 1 - \prod_{i=n}^{n} P(X_i > z)$

$$= 1 - \{1 - F_X(z)\}^n$$

$$f_Z(z) = \frac{\partial}{\partial z} F_Z(z) = n\{1 - F_X(z)\}^{n-1} f_X(z)$$

Ex 3. X has pdf $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Find the distribution of $Y = 8X^3$.

Answer: First, note that $0 < y < 8$.

$$F_Y(y) = P(Y \leq y) = P\left(8X^3 < y\right) = P\left\{X \leq \left(\frac{y}{8}\right)^{1/3}\right\} = \int_0^{\left(\frac{y}{8}\right)^{1/3}} 2x\,dx = \frac{1}{4}y^{2/3}, \quad 0 < y < 8$$

$$f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \begin{cases} \frac{1}{6\sqrt[3]{y}}, & 0 < y < 8 \\ 0, & \text{otherwise} \end{cases}$$

PDF Method

Basically same as the cdf method, except that the pdf is used instead of the cdf. Suppose $Y = T(X)$ with a transformation, T: 1-1 transformation from $\mathscr{A}$ onto $\mathscr{B}$. Then Y will have a pdf:

$$f_Y(y) = \begin{cases} f_X\left\{T^{-1}(y)\right\} |J_{T^{-1}(y)}|, & y \in \mathscr{B} \\ 0, & \text{otherwise} \end{cases}$$

Ex 4. X has pdf $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Find the distribution of $Y = 8X^3$.

Answer: First, note that $0 < y < 8$.

$$T : y = 8x^3$$

$$T^{-1} : x = \left(\frac{y}{8}\right)^{1/3} = \frac{1}{2}y^{1/3}$$

$$f_X\left\{T^{-1}(y)\right\} = 2 \cdot \frac{1}{2}y^{1/3} = y^{1/3}$$

$$|J_{T^{-1}(y)}| = \left|\frac{\partial}{\partial y}\left(\frac{1}{2}y^{1/3}\right)\right| = \frac{1}{6}y^{-2/3}$$

$$\therefore\ f_Y(y) = f_X\left\{T^{-1}(y)\right\}|J_{T^{-1}(y)}| = \frac{1}{6}y^{-1/3},\ 0<y<8$$

Ex 5. $X \sim uniform\,(0,1)$. Find the distribution of $Y = -2\ln X$.

Answer: First, note that $y > 0$.

$$\begin{aligned}
&T: y = -2\ln x \qquad : \text{1-1 from } \mathscr{A} = (0,1) \text{ onto } \{y : y>0\}\\
&T^{-1}: x = e^{-\frac{1}{2}y}\\
&f_X\left\{T^{-1}(y)\right\} = 1, \quad y>0\\
&|J_{T^{-1}(y)}| = \left|\frac{\partial}{\partial y}\left(e^{-\frac{1}{2}y}\right)\right| = \frac{1}{2}e^{-\frac{1}{2}y}
\end{aligned}$$

$$\therefore\ f_Y(y) = f_X\left\{T^{-1}(y)\right\}|J_{T^{-1}(y)}| = \frac{1}{2}e^{-\frac{1}{2}y},\ y>0$$

i.e., $Y \sim \chi^2_{(df=2)}$.

Ex 6. Let $X_1 \sim \Gamma(\alpha,1)$, $X_2 \sim \Gamma(\beta,1)$ and X_1, X_2: independent. Show

1. distribution of $\dfrac{X_1}{X_1+X_2}$ is $\beta(\alpha,\beta)$
2. $X_1 + X_2 \sim \Gamma(\alpha+\beta,1)$
3. X_1+X_2 and $\dfrac{X_1}{X_1+X_2}$ are independent.

Proof:

$$T: \begin{cases} y_1 = x_1+x_2 \\ y_2 = \frac{x_1}{x_1+x_2}\end{cases} \quad : \text{1-1 from } \mathscr{A} = (0,\infty)\times(0,\infty) \text{ onto } \{y_1y_2>0,\ y_1-y_1y_2>0\}$$

$$T^{-1}: \begin{cases} x_1 = y_1y_2 \\ x_2 = y_1 - y_1y_2\end{cases} \qquad y_1>0,\ 0<y_2<1$$

$$|J_{T^{-1}}| = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1\end{vmatrix} = |y_1| = y_1$$

$$f(x_1,x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}x_1^{\alpha-1}x_2^{\beta-1}e^{-x_1}e^{-x_2}$$

$$\begin{aligned}
\therefore\quad f(y_1,y_2) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(y_1y_2)^{\alpha-1}\left\{y_1(1-y_2)\right\}^{\beta-1}e^{-y_1}\cdot y_1, \quad y_1>0,\ 0<y_2<1\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}y_1^{\alpha+\beta-1}e^{-y_1}y_2^{\alpha-1}(1-y_2)^{\beta-1}\\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y_2^{\alpha-1}(1-y_2)^{\beta-1}I_{(0,1)}(y_2)\cdot\frac{1}{\Gamma(\alpha+\beta)}y_1^{\alpha+\beta-1}e^{-y_1}I_{(0,\infty)}(y_1)\\
&= f(y_2)\cdot f(y_1), \quad Y_2 \sim \beta(\alpha,\beta) \text{ and } Y_1 \sim \Gamma(\alpha+\beta,1)
\end{aligned}$$

Ex 7. Let $X_1 \sim \Gamma(\alpha, \theta)$, $X_2 \sim \Gamma(\beta, \theta)$ and X_1, X_2: independent. Show

1. distribution of $\dfrac{X_1}{X_1 + X_2}$ is $\beta(\alpha, \beta)$
2. $X_1 + X_2 \sim \Gamma(\alpha + \beta, \theta)$
3. $X_1 + X_2$ and $\dfrac{X_1}{X_1 + X_2}$ are independent.

Proof:

$$T : \begin{cases} y_1 = x_1 + x_2 \\ y_2 = \frac{x_1}{x_1+x_2} \end{cases} \quad : \text{1-1 from } \mathscr{A} = (0,\infty) \times (0,\infty) \text{ onto } \{y_1 y_2 > 0,\ y_1 - y_1 y_2 > 0\}$$

$$T^{-1} : \begin{cases} x_1 = y_1 y_2 \\ x_2 = y_1 - y_1 y_2 \end{cases} \qquad y_1 > 0,\ 0 < y_2 < 1$$

$$|J_{T^{-1}}| = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = |y_1| = y_1$$

$$f(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^\alpha \theta^\beta} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1/\theta} e^{-x_2/\theta}$$

$$\begin{aligned} \therefore \qquad f(y_1, y_2) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^\alpha \theta^\beta} (y_1 y_2)^{\alpha-1} \{y_1(1-y_2)\}^{\beta-1} e^{-y_1/\theta} \cdot y_1, \quad y_1 > 0,\ 0 < y_2 < 1 \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^\alpha \theta^\beta} y_1^{\alpha+\beta-1} e^{-y_1/\theta} y_2^{\alpha-1} (1-y_2)^{\beta-1} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1}(1-y_2)^{\beta-1} I_{(0,1)}(y_2) \cdot \frac{1}{\Gamma(\alpha+\beta)\theta^{\alpha+\beta}} y_1^{\alpha+\beta-1} e^{-y_1/\theta} I_{(0,\infty)}(y_1) \\ &= f(y_2) \cdot f(y_1), \quad Y_2 \sim \beta(\alpha, \beta) \text{ and } Y_1 \sim \Gamma(\alpha+\beta, \theta) \end{aligned}$$

Corollary $X_1 \sim \chi^2(\alpha)$, $X_2 \sim \chi^2(\beta)$ and X_1, X_2: independent. Let $Y_1 = \dfrac{X_1}{X_1 + X_2}$ and $Y_2 = X_1 + X_2$. Then, $Y_1 \sim \beta\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$, $Y_2 \sim \Gamma\left(\frac{\alpha+\beta}{2}, 2\right) \sim \chi^2(\alpha+\beta)$ and Y_1, Y_2 are independent.

Ex 8. Let X_1, X_2: iid exponential with λ, i.e., $\Gamma\left(1, \frac{1}{\lambda}\right)$. Find the distribution of $Y_1 = X_1 - X_2$.

Answer:

$$T:\begin{cases} y_1 = x_1 - x_2 \\ y_2 = x_2 \end{cases} \quad : \text{ 1-1 from } \mathscr{A} = (0,\infty)\times(0,\infty) \text{ onto } \{y_1+y_2>0,\, y_2>0\}$$

$$T^{-1}:\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_2 \end{cases} \quad y_1+y_2>0,\, y_2>0$$

$$|J_{T^{-1}}| = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f(x_1,x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}, \quad x_1>0,\, x_2>0$$

$$\therefore \quad f(y_1,y_2) = \lambda^2 e^{-\lambda(y_1+2y_2)}, \quad y_1+y_2>0,\, y_2>0$$

$$\begin{aligned} \therefore \quad f_1(y_1) &= \int_{-\infty}^{\infty} f(y_1,y_2)dy_2 \\ &= \begin{cases} \displaystyle\int_0^{\infty} \lambda^2 e^{-\lambda(y_1+2y_2)}dy_2, & y_1>0 \\ \displaystyle\int_{-y_1}^{\infty} \lambda^2 e^{-\lambda(y_1+2y_2)}dy_2, & y_1\le 0 \end{cases} \\ &= \begin{cases} \frac{\lambda}{2}e^{-\lambda y_1}, & y_1>0 \\ \frac{\lambda}{2}e^{\lambda y_1}, & y_1\le 0 \end{cases} \\ &= \frac{\lambda}{2}e^{-\lambda|y_1|}, \quad -\infty<y_1<\infty \end{aligned}$$

Y_1 is called the double-exponential distribution or Lapace distribution.

Ex 9. Box-Muller transformation: Let X_1, X_2: iid uniform (0,1). Then

$$\begin{aligned} Y_1 &= \sqrt{-2\ln X_1}\cos(2\pi X_2) \\ Y_2 &= \sqrt{-2\ln X_1}\sin(2\pi X_2) \\ \Rightarrow \quad & Y_1, Y_2 \sim iid\, N(0,1) \end{aligned}$$

Proof:

$$T:\begin{cases} y_1=\sqrt{-2\ln x_1}\cos(2\pi x_2) \\ y_2=\sqrt{-2\ln x_1}\sin(2\pi x_2)\end{cases} \quad : \text{ 1-1 from } \mathscr{A} = (0,1)\times(0,1) \text{ onto } \Re^2$$

$$T^{-1}:\begin{cases} x_1=\exp\left(-\frac{y_1^2+y_2^2}{2}\right) \\ x_2=\frac{1}{2\pi}\tan^{-1}\left(\frac{y_2}{y_1}\right)\end{cases}$$

$$|J_{T^{-1}}| = \begin{vmatrix} -x_1 e^{-\frac{y_1^2+y_2^2}{2}} & -x_2 e^{-\frac{y_1^2+y_2^2}{2}} \\ \frac{1}{2\pi}\frac{-y_2/y_1^2}{1+(y_2/y_1)^2} & \frac{1}{2\pi}\frac{1/y_1}{1+(y_2/y_1)^2}\end{vmatrix} = \frac{1}{2\pi}e^{-\frac{y_1^2+y_2^2}{2}}$$

$$f(x_1,x_2)=I_{(0,1)}(x_1)\cdot I_{(0,1)}(x_2)$$

$$\therefore \quad f(y_1,y_2)=\frac{1}{2\pi}e^{-\frac{y_1^2+y_2^2}{2}}, \quad -\infty<y_1,y_2<\infty$$

$$\therefore \quad Y_1,Y_2\sim iid\, N(0,1)$$

Ex 10. X_1, X_2: iid $N(0,1)$. Then $Y=\frac{X_1}{X_2}\sim\mathscr{C}(0,1)$, i.e., Cauchy with pdf $\frac{1}{\pi}\frac{1}{1+y^2}$, $-\infty<y<\infty$.
Proof:

$$T:\begin{cases} y_1=\frac{x_1}{x_2} \\ y_2=x_2\end{cases} \quad : \text{ 1-1 from } \Re^2\setminus\{x_2=0\} \text{ onto } \Re^2$$

$$T^{-1}:\begin{cases} x_1=y_1y_2 \\ x_2=y_2\end{cases} \quad y_1+y_2>0,\ y_2>0$$

$$|J_{T^{-1}}| = \begin{vmatrix} y_2 & y_1 \\ 0 & 1\end{vmatrix} = |y_2|$$

$$f(x_1,x_2)=\frac{1}{2\pi}e^{-\frac{x_1^2+x_2^2}{2}}, \quad x_1,x_2\in\Re$$

$$\therefore \quad f(y_1,y_2)=\frac{1}{2\pi}|y_2|e^{-\frac{y_1^2y_2^2+y_2^2}{2}}, \quad y_1,y_2\in\Re$$

$$\therefore \quad f_1(y_1)=\int_{-\infty}^{\infty}f(y_1,y_2)dy_2=\frac{1}{2\pi}\int_{-\infty}^{\infty}|y_2|e^{-\frac{1+y_1^2}{2}y_2^2}dy_2$$

$$=\frac{1}{\pi}\int_0^{\infty}|y_2|e^{-(1+y_1^2)t}dt=\frac{1}{\pi}\frac{1}{1+y_1^2}, \quad y_1\in\Re$$

One note: General form of the Cauchy pdf:

$$X\sim\mathscr{C}(\mu,\sigma) \qquad \to \qquad f(x)=\frac{1}{\sigma}\frac{1}{\pi}\frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x\in\Re$$

Selected Problems

1. Let X have the pdf $f(x) = 4x^3$, $0 < x < 1$. Find the pdf of $Y = X^2$.
2. Let X have the pdf $f(x) = xe^{-x^2/2}$, $0 < x < \infty$. Find the pdf of $Y = X^2$.
3. Let X have a $\Gamma(\alpha = 3, \theta = 2)$. Determine the pdf of $Y = \sqrt{X}$.
4. Let X have the pdf $f(x) = 2x$, $0 < x < 1$.
   (a) Find the cdf of X.
   (b) Deescribe how an observation of X can be simulated.
   (c) Simulate 10 observations of X.
5. Let X have the pdf $f(x) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$. Find the distribution of $Y = -2\theta \ln X$.
6. Let X have a *logistic distribution* with the pdf $f(x) = \dfrac{e^{-x}}{(1+e^{-x})^2}$, $-\infty < x < \infty$. Show that $Y = \dfrac{1}{1+e^{-X}}$ has a uniform (0, 1) distribution.
7. Suppose the lifetime (in years) of a product is $Y = 5X^{0.7}$, where X has an exponential distribution with mean 1. Find the cdf and pdf of Y.