Chapter 4. Application to Statistical Inference

Objectives

- Basic Theoretical Results
- Confidence Interval and Hypotheses Testing
- Tests on One Parameter & Two Parameters
- Simple Linear Regression

Applications

Definition 1. *t-distribution*

$$T=\frac{Z}{\sqrt{U/r}},\quad \textit{where } Z\sim N(0,1),\, U\sim\chi^2(r),\, \textit{and } Z\&U:\textit{indep.}$$

$\Rightarrow\quad T\sim t(r)$, *i.e., T has (student's) t-distribution with df=r.*

Application: $X_1,\ldots,X_n$: random samples from $N\left(\mu,\sigma^2\right)$, then

$$\frac{\bar{X}-\mu}{\sqrt{S/n}}\sim t(n-1)$$

Proof.

$$\bar{X}\sim N\left(\mu,\frac{\sigma^2}{n}\right),\quad \frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$$

$$\Rightarrow\quad \frac{\bar{X}-\mu}{\sqrt{S/n}}=\frac{\frac{\bar{X}-\mu}{\sigma\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}}=\frac{Z}{\sqrt{U/r}}\sim t(n-1)$$

$\square$

The $100(1-\alpha)$ % confidence interval for μ is

$$P\left\{\bar{X}-t_{\alpha/2}\frac{s}{\sqrt{n}}\leq\mu\leq\bar{X}+t_{\alpha/2}\frac{s}{\sqrt{n}}\right\}=1-\alpha,$$

where $t_{\alpha/2}$ is the upper $\alpha/2$ quantile of the t-distribution with $df=(n-1)$.
Note also that $T\sim -T$, i.e., T is symmetric about zero.

$$\because\quad -T=\frac{-Z}{\sqrt{U/r}},\quad \text{where } -Z\sim N(0,1),\, U\sim\chi^2(r),\, \text{and } -Z\&U:\textit{indep.}$$

One note: pdf of $t(r)$ is derived below.

$$T = \frac{Z}{\sqrt{U/r}}$$

$$T : \begin{cases} t = \frac{z}{\sqrt{u/r}} \\ v = u \end{cases} \quad \text{1-1 transformation from } (-\infty,\infty)\times(0,\infty) \text{ onto } (-\infty,\infty)\times(0,\infty)$$

$$T^{-1} : \begin{cases} z = t\sqrt{\frac{v}{r}} \\ u = v \end{cases}$$

$$|J_{T^{-1}}| = \det \begin{vmatrix} \sqrt{\frac{v}{r}} & \frac{t}{\sqrt{r}}\frac{1}{2\sqrt{v}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{v}{r}}$$

$$f_{Z,U}(z,u) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \cdot \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} u^{\frac{r}{2}-1} e^{-\frac{u}{2}}, \quad -\infty < z < \infty,\ u > 0$$

$$\begin{aligned} g_{T,V}(t,v) &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} \cdot e^{-\left(\frac{v}{r}t^2+v\right)/2} v^{\frac{r}{2}-1} \cdot \sqrt{\frac{v}{r}} \\ &= \frac{1}{\sqrt{2\pi}\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}} \sqrt{r}} \cdot e^{-\frac{t^2+r}{2r}v} \cdot v^{\frac{r+1}{2}-1} \end{aligned}$$

$$\begin{aligned} \therefore \quad g_T(t) &= \frac{1}{\sqrt{2\pi}\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}} \sqrt{r}} \cdot \Gamma\left(\frac{r+1}{2}\right) \cdot \left(\frac{2r}{r+t^2}\right)^{\frac{r+1}{2}} \int_0^\infty v^{\frac{r+1}{2}-1} e^{-\frac{t^2+r}{2r}v} dv \\ &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}}, \quad -\infty < t < \infty \end{aligned}$$

Definition 2. *F-distribution*

$$F = \frac{U/r_1}{V/r_2} \sim F(r_1, r_2), \quad \textit{where } U \sim \chi^2(r_1),\ V \sim \chi^2(r_2), \textit{ and } U\&V : \textit{indep.}$$

$\Rightarrow \quad F \sim F(r_1, r_2)$, *i.e., F has an F-distribution with df's=*r_1, r_2.

Application: $X_1, \ldots, X_{n_1}$: random samples from $N\left(\mu_1, \sigma_1^2\right)$, and $Y_1, \ldots, Y_{n_2}$: random samples from $N\left(\mu_2, \sigma_2^2\right)$, then

$$\frac{\sigma_2^2}{\sigma_1^2}\frac{S_1^2}{S_2^2} = \frac{\frac{(n_1-1)S_1^2}{\sigma_1^2}/(n_1-1)}{\frac{(n_2-1)S_2^2}{\sigma_2^2}/(n_2-1)} = \frac{\frac{(n_1-1)S_1^2}{(n_1-1)}/\sigma_1^2}{\frac{(n_2-1)S_2^2}{(n_2-1)}/\sigma_2^2} \sim F(n_1-1, n_2-1)$$

The $100(1-\alpha)$ % confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$ is

$$P\left\{\frac{S_2^2}{S_1^2}F_{(1-\alpha/2;\,n_1-1,n_2-1)} \le \frac{\sigma_2^2}{\sigma_1^2} \le \frac{S_2^2}{S_1^2}F_{(\alpha/2;\,n_1-1,n_2-1)}\right\} = 1-\alpha,$$

where $F_{(\alpha/2;\, n_1-1, n_2-1)}$ is the upper $\alpha/2$ quantile of the F-distribution with $df = (n_1 - 1, n_2 - 1)$.

- If $F \sim F(df_1, df_2)$ then $\dfrac{1}{F} \sim F(df_2, df_1)$
- $F_{(1-\alpha/2;\, df_1, df_2)} = \dfrac{1}{F_{(\alpha/2;\, df_2, df_1)}}$
- If $T \sim t(r)$, then $T^2 \sim F_{1,r}$

Proof.

$$
\begin{aligned}
&F = \frac{U/df_1}{V/df_2} \sim F(df_1, df_2) \quad \Rightarrow \quad \frac{1}{F} = \frac{V/df_2}{U/df_1} \sim F(df_2, df_1) \\
&P\left\{F \le F_{(1-\alpha/2;\, df_1, df_2)}\right\} = 1 - \frac{\alpha}{2} \\
&P\left\{F \le \frac{1}{F_{(\alpha/2;\, df_2, df_1)}}\right\} = P\left\{\frac{1}{F} \ge F_{(\alpha/2;\, df_2, df_1)}\right\} = 1 - \frac{\alpha}{2} \\
&\text{Recall that } T = \frac{Z}{\sqrt{U/r}}, \quad \text{where } Z \sim N(0,1),\ U \sim \chi^2(r), \text{ and } Z\&U : \textit{indep.} \\
&\Rightarrow \quad T^2 = \frac{Z^2}{U/r}, \quad \text{where } Z^2 \sim \chi^2(1),\ U \sim \chi^2(r), \text{ and } Z^2\&U : \textit{indep.} \\
&\Rightarrow \quad T^2 \sim F_{1,r}
\end{aligned}
$$

□

Another note: Derivation of the pdf of $F(r_1, r_2)$.

$$
\begin{aligned}
&F = \frac{U/r_1}{V/r_2} \\
&T : \begin{cases} f = \frac{u/r_1}{v/r_2} \\ z = v \end{cases} \quad \text{1-1 transformation from } (0,\infty)\times(0,\infty) \text{ onto } (0,\infty)\times(0,\infty) \\
&T^{-1} : \begin{cases} u = \frac{r_1}{r_2} f z \\ v = z \end{cases} \\
&|J_{T^{-1}}| = \det \begin{vmatrix} \frac{r_1}{r_2} z & \frac{r_1}{r_2} f \\ 0 & 1 \end{vmatrix} = \frac{r_1}{r_2} z \\
&f_{U,V}(u,v) = \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{\frac{r_1+r_2}{2}}} u^{\frac{r_1}{2}-1} v^{\frac{r_2}{2}-1} e^{-\frac{u+v}{2}}, \quad u > 0,\ v > 0 \\
&g_{F,Z}(f,z) = \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{\frac{r_1+r_2}{2}}} \left(\frac{r_1}{r_2} f z\right)^{\frac{r_1}{2}-1} z^{\frac{r_2}{2}-1} e^{-\frac{\frac{r_1}{r_2} f + 1}{2} z} \cdot z \frac{r_1}{r_2}
\end{aligned}
$$

$$\therefore \quad g_F(f) = \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)2^{\frac{r_1+r_2}{2}}}\left(\frac{r_1}{r_2}f\right)^{\frac{r_1}{2}-1}\left(\frac{r_1}{r_2}\right)\int_0^\infty z^{\frac{r_1+r_2}{2}-1}e^{-\frac{r_1 f+r_2}{2r_2}z}dz$$

$$= \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)2^{\frac{r_1+r_2}{2}}}\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}-1} f^{\frac{r_1}{2}-1}\left(\frac{r_1}{r_2}\right)\Gamma\left(\frac{r_1+r_2}{2}\right)\left(\frac{2r_2}{r_1 f+r_2}\right)^{\frac{r_1+r_2}{2}}$$

$$= \frac{\Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)}\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} \cdot \frac{f^{\frac{r_1}{2}-1}}{\left(1+\frac{r_1}{r_2}f\right)^{\frac{r_1+r_2}{2}}}, \quad f > 0$$

Ex 1. $X \sim N(\mu, \sigma^2)$, and $n = 20$, $\bar{X} = 507.50$, $s = 89.75$. Find a 90% confidence interval for μ.
Answer:

$$P\left\{\bar{X} - t_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2}\frac{s}{\sqrt{n}}\right\} = 1-\alpha,$$

Also, $t_{19;\,0.05} = 1.7291$, a 90% confidence interval for μ is

$$507.50 \pm 1.7291 \cdot \frac{89.72}{20} = 507.50 \pm 34.70 = (472.80,\ 542.20)$$

Here is R printout for your information.

```
> qt(0.95,19); qt(0.05,19)
[1] 1.729133
[1] -1.729133
> mean(x); sd(x)
[1] 507.5
[1] 89.75082
> 507.50+(c(-1.7291,1.7291))*89.72/(sqrt(20))
[1] 472.8108 542.1892
> x <- c(481,537,513,583,453,510,570,500,457,555,618,327,350,643,499,421,505,637,599,392)
> t.test(x,conf.level=0.90)

        One Sample t-test

data:  x
t = 25.2879, df = 19, p-value = 4.311e-16
alternative hypothesis: true mean is not equal to 0
90 percent confidence interval:
 472.7982 542.2018
sample estimates:
mean of x
    507.5
```

Ex 2. $X \sim N(\mu, \sigma^2)$, and $n = 13$, $s^2 = 10.70\,(i.e., s = 3.27)$. Find a 90% confidence interval for σ^2.
Answer:

$$P\left\{\frac{(n-1)S^2}{\chi^2_{n-1,\,\alpha/2}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{n-1,\,1-\alpha/2}}\right\} = 1-\alpha,$$

Also, $\chi^2_{12;\,0.05} = 21.03$, and $\chi^2_{12;\,0.95} = 5.226$, so a 90% confidence interval for σ^2 is

$$\left[\frac{(12)(10.70)}{21.03},\quad \frac{(12)(10.70)}{5.226}\right] = (6.11,\ 24.57)$$

A 90% confidence interval for σ is $(\sqrt{6.11},\ \sqrt{24.57}) = (2.47,\ 4.96)$. Here is relevant R printout.

```
> qchisq(0.05,12); qchisq(0.95,12)
[1] 5.226029
[1] 21.02607
```

Ex 3. Suppose $n_1 = 13$ and $n_2 = 9$ random samples are taken from two separate normal distributions with different means and different variances. Let $s_1^2 = 10.70$, $s_2^2 = 4.59$. Find a 98% confidence interval for the ratio of the two variances, i.e., $\dfrac{\sigma_1^2}{\sigma_2^2}$.

Answer:

$$P\left\{\frac{S_1^2}{S_2^2}F_{(1-\alpha/2;\,n_2-1,n_1-1)} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_2^2}{S_1^2}F_{(\alpha/2;\,n_2-1,n_1-1)}\right\} = 1-\alpha,$$

Also, $F_{1-0.99;\,8,12} = 0.1765$, and $F_{0.99;\,8,12} = 4.50$, so a 98% confidence interval for $\dfrac{\sigma_1^2}{\sigma_2^2}$ is

$$\left[0.1765\cdot\frac{10.70}{4.59},\quad 4.50\cdot\frac{10.70}{4.59}\right] = (0.41,\ 10.49)$$

Also, a 90% confidence interval for $\dfrac{\sigma_1}{\sigma_2}$ is $(\sqrt{0.41},\ \sqrt{10.49}) = (0.64,\ 3.24)$. Here is R printout for your information.

```
> qf(0.99,8,12); qf(0.01,8,12)
[1] 4.499365
[1] 0.176469
> 1/(qf(0.99,12,8))      # Famous property of F
[1] 0.176469
```

Application: Let $X_1,\ldots,X_n$: iid $N\left(\mu_X,\sigma^2\right)$, and $Y_1,\ldots,Y_m$: iid $N\left(\mu_Y,\sigma^2\right)$, then
$S_p^2 = \dfrac{1}{n+m-2}\left\{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2 + \sum_{i=1}^{m}\left(Y_i-\bar{Y}\right)^2\right\} \sim \chi^2_{(n+m-2)}$ and $E\left(S_p^2\right) = \sigma^2$

Also, a $100\,(1-\alpha)\,\%$ confidence interval for $\mu_X-\mu_Y$ is

$$P\left\{\left(\bar{X}-\bar{Y}\right) - t_{\alpha/2}S_p\sqrt{\frac{1}{n}+\frac{1}{m}} \le \mu_X-\mu_Y \le \left(\bar{X}-\bar{Y}\right) + t_{\alpha/2}S_p\sqrt{\frac{1}{n}+\frac{1}{m}}\right\} = 1-\alpha,$$

Proof.

$$\frac{(n-1)S_X^2}{\sigma^2} \sim \chi^2(n-1) \quad \text{and} \quad \frac{(m-1)S_Y^2}{\sigma^2} \sim \chi^2(m-1) \quad \text{and indep.}$$

$$\Rightarrow \quad \frac{(n-1)S_X^2 + (m-1)S_Y^2}{\sigma^2} \sim \chi^2(n+m-2)$$

$$\Rightarrow \quad T = \frac{\frac{\bar{X}-\bar{Y}-(\mu_X-\mu_Y)}{\sqrt{\sigma^2/n+\sigma^2/m}}}{\sqrt{\frac{(n-1)S_X^2+(m-1)S_Y^2}{\sigma^2}/(n+m-2)}} \quad \sim \quad \frac{N(0,1)}{\sqrt{\chi^2_{(m+m-2)}/(n+m-2)}} \sim t_{(n+m-2)}$$

$$\Rightarrow \quad T = \frac{\bar{X}-\bar{Y}-(\mu_X-\mu_Y)}{\sqrt{\frac{(n-1)S_X^2+(m-1)S_Y^2}{n+m-2}\left(\frac{1}{n}+\frac{1}{m}\right)}} \quad \sim \quad t_{(n+m-2)}$$

□

Ex 4. Suppose $n_1 = 9, \bar{X}_1 = 81.31, s_1^2 = 60.76$ and $n_2 = 15, \bar{X}_2 = 78.61, s_2^2 = 48.24$. Two sets of random samples are taken from two separate normal distributions with different means but having common variances. Find a 95% confidence interval for the difference of the two means, i.e., $\mu_X - \mu_Y$.
Answer:

$$(81.31 - 78.61) \pm 2.074 \cdot \sqrt{\frac{(8)(60.76)+(14)(48.24)}{22}}\sqrt{\frac{1}{9}+\frac{1}{15}},$$

where $t_{0.025;\,22} = 2.074$, so a 95% confidence interval for $\mu_X - \mu_Y$ is (−3.65, 9.05).

Statistical Hypothesis Testing

Ex 5. *One-sample problem:* After years of research and investment, "new" batteries have been developed. Existing batteries run for about 16 hours when used continuously. A random sample of 9 new batteries was tested and the result: `12 24 18 30 19 17 25 27 21`. What do you think? Is there any evidence of improvement?

```
x <- c(12, 24, 18, 30, 19, 17, 25, 27, 21)
quantile(x)
  0%  25%  50%  75% 100%
  12   18   21   25   30
quantile(x,type=6)            #This agrees with Minitab & SPSS rule of (n+1)/4th
  0%  25%  50%  75% 100%
12.0 17.5 21.0 26.0 30.0
summary(x)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  12.00   18.00   21.00   21.44   25.00   30.00
mean(x); var(x); sd(x)
[1] 21.44444
[1] 31.27778
[1] 5.592654
boxplot(x); points(mean(x), cex=4, pch=16,col=("red"))
```

```
boxplot(x, horizontal=T)
qqnorm(x)
qqline(x)
plot(density(x)); rug(x)
shapiro.test(x)
        Shapiro-Wilk normality test
data:  x
W = 0.9853, p-value = 0.9858

t.test(x, mu=16)              # t.test(x,mu=16,alternative=c("greater"))
        One Sample t-test
data:  x
t = 2.9205, df = 8, p-value = 0.01928    # We reject H0 at 0.05. True mean is significantly > 16.
alternative hypothesis: true mean is not equal to 16
95 percent confidence interval:
 17.14555 25.74334
sample estimates:
mean of x
 21.44444
```

- 95% confidence interval (c.i.) for the true mean μ is calculated below, assuming that data are from a normal population.
$$\bar{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} = 21.44 \pm \left(2.306 \times \frac{5.59}{\sqrt{9}}\right) = (17.14,\, 25.74)$$

- Statistical hypothesis test:

$$H_0 : \mu = 16 \text{ (i.e., “new” batteries are not different from “existing” batteries.)}$$
$$H_1 : \mu > 16 \text{ (i.e., “new” batteries last significantly longer than “existing” batteries.)}$$
$$\text{test statistic } t = \frac{\bar{x} - 16}{s/\sqrt{n}} = \frac{21.44 - 16}{5.59/\sqrt{9}} = 2.92 \quad \sim t_{df=8}$$
$$p - \text{value} = 0.0096$$

We reject H_0 since p-value is much less than 0.05, and we conclude that the “new” batteries last significantly longer.

Statistical Hypothesis Test: Four Steps

1. Write two competing hypotheses: H_0 (null hypothesis) vs. H_1 (alternative/research hypothesis)
2. Compute an appropriate test statistic (T.S.)
3. Find the p-value of a T.S.
4. **Reject H_0 if p-value is less than α (= usually 0.05)**. State your conclusion in “plain” terms.

Facts:

1. See how to write two hypotheses:

H_0	H_1
He is NOT guilty.	He is guilty.
New product is NOT different from the existing one.	New product is significantly different from the existing one.
Stress affects men and women equally.	Stress affects women significantly more than women.

2. p-value = probability of the null hypothesis being true (assuming that it is true).
3. We reject H_0 if p-value is less than α (= usually 0.05). p-value tells us "how extreme" the observed test statistic is assuming the null hypothesis is true. For example, it calculates the chances that he is NOT guilty (from all the evidences at the crime scene) if he was indeed NOT guilty.

1-tailed vs. 2-tailed hypothesis:

H_0	H_1
New product is NOT different from the existing one.	New product is significantly better than the existing one.
	New product is different from the existing one.
Stress affects men and women equally.	Stress affects women significantly more than women.
	Stress affects men and women differently.

Four Possibilities:

	when H_0 is true	when H_1 is true
H_0 is NOT rejected.	OK	Type II error
H_0 is rejected.	Type I error	OK

$$\alpha = P(\text{committing Type I error})$$
$$\beta = P(\text{committing Type II error})$$
$$1 - \beta = \textit{power} = P(\text{correctly rejecting } H_0 \text{ when } H_1 \text{ is actually true})$$

Examples of the "Type I Errors" are:

- Incarcerating an innocent person
- Promoting a so-so product

Examples of the "Type II Errors" are:

- Person who actually committed the crime was found not guilty
- Discarding a truly better product

Another Look:

	when you do NOT have the disease	when you DO have the disease
You're tested −	OK = "Specificity"	Type II error = false negative
You're tested +	Type I error = false positive	OK = "Sensitivity"

Test on a single population mean (μ):

$$H_0: \mu = \mu_0 \qquad vs. \qquad \begin{array}{ll} H_1: \mu > \mu_0 & \text{(1-tailed hypothesis)} \\ H_1: \mu < \mu_0 & \text{(1-tailed hypothesis)} \\ H_1: \mu \neq \mu_0 & \text{(2-tailed hypothesis)} \end{array}$$

$$\text{T.S. } t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad \sim t_{df=(n-1)}$$

$$p-\text{value} = P(t > T.S.), \quad p-\text{value} = P(t > T.S.), \quad p-\text{value} = 2P(t > |T.S.|)$$

Test on two population means (μ_1, μ_2):

$$H_0: \mu_1 = \mu_2 \qquad vs. \qquad \begin{array}{ll} H_1: \mu_1 > \mu_2 & \text{(1-tailed hypothesis)} \\ H_1: \mu_1 < \mu_2 & \text{(1-tailed hypothesis)} \\ H_1: \mu_1 \neq \mu_2 & \text{(2-tailed hypothesis)} \end{array}$$

$$\text{T.S. (pooled } t\text{-test) } T = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}\left(\frac{1}{n} + \frac{1}{m}\right)}} \quad \sim \quad t_{(n+m-2)}$$

Ex 6. *Two-sample problem:* Compare the following two groups. Is there a significant "mean" difference?

Group 1	16.4	29.4	37.1	23.0	24.1	24.5	16.4	29.1	36.7	28.7	30.2	21.8	37.1	20.3	28.3
Group 2	22.2	34.8	42.1	32.9	26.4	30.6	32.9	37.5	18.4	27.5	45.5	34.0	45.5	24.5	28.7

```
x <- c(16.4, 29.4, 37.1, 23.0, 24.1, 24.5, 16.4, 29.1, 36.7, 28.7, 30.2, 21.8, 37.1, 20.3, 28.3)
y <- c(22.2, 34.8, 42.1, 32.9, 26.4, 30.6, 32.9, 37.5, 18.4, 27.5, 45.5, 34.0, 45.5, 24.5, 28.7)
boxplot(x, y)
summary(x); summary(y)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  16.40   22.40   28.30   26.87   29.80   37.10
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  18.40   26.95   32.90   32.23   36.15   45.50
mean(x); mean(y)
[1] 26.87333
[1] 32.23333
sd(x); sd(y)
[1] 6.814739
[1] 8.067188
var.test(x, y)

        F test to compare two variances

data:  x and y
F = 0.7136, num df = 14, denom df = 14, p-value = 0.5361
    # We do NOT reject H0 at 0.05. Two variances are NOT significantly different.
```

```
alternative hypothesis: true ratio of variances is not equal to 1
95 percent confidence interval:
 0.2395762 2.1255163
sample estimates:
ratio of variances
         0.7135987

t.test(x, y, var.equal=TRUE, paired=FALSE)

        Two Sample t-test

data:  x and y
t = -1.9658, df = 28, p-value = 0.05931
    # We do NOT reject H0 at 0.05. Two means  are NOT significantly different.
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -10.9453115   0.2253115
sample estimates:
mean of x mean of y
 26.87333  32.23333
```

Test on two population means (μ_1, μ_2)**:**

$$\begin{aligned} H_0: \mu_1 = \mu_2 \qquad vs. \qquad & H_1: \mu_1 > \mu_2 \quad \text{(1-tailed hypothesis)} \\ & H_1: \mu_1 < \mu_2 \quad \text{(1-tailed hypothesis)} \\ & H_1: \mu_1 \neq \mu_2 \quad \text{(2-tailed hypothesis)} \end{aligned}$$

$$\text{T.S. (Welch } t\text{-test) } T = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \quad \sim \quad t_{df'}, \text{ where } df' = \frac{\left\{\left(s_1^2/n_1\right) + \left(s_2^2/n_2\right)\right\}^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1-1} + \frac{\left(s_2^2/n_2\right)^2}{n_2-1}}.$$

Test on "paired" (i.e., "matched") population mean (μ_d)**:**

$$\begin{aligned} H_0: \mu_d = 0 \qquad vs. \qquad & H_1: \mu_d > 0 \quad \text{(1-tailed hypothesis)} \\ & H_1: \mu_d < 0 \quad \text{(1-tailed hypothesis)} \\ & H_1: \mu_d \neq 0 \quad \text{(2-tailed hypothesis)} \end{aligned}$$

$$\text{T.S. } t = \frac{\bar{x}_d}{s_d/\sqrt{n_d}} \quad \sim t_{df=(n_d-1)}$$

Ex 7. *Paired-sample problem:* In a small town, there are two garages, and customers are asked to bring two estimates every time they're involved in an accident. Insurance adjusters are concerned about the high estimates they are receiving from garage 1 for auto repairs compared to garage 2. To verify their suspicions each of 15 cars recently involved in car accidents was taken to both garages for separate estimates of repair costs. Is there any evidence to justify the suspicion?

Garage I	760	1020	950	130	300	630	530	620	220	480	1130	1210	690	760	840
Garage II	730	910	840	150	270	580	490	530	200	420	1100	1100	610	670	750

```
x <- c(760, 1020, 950, 130, 300, 630, 530, 620, 220, 480, 1130, 1210, 690, 760, 840)
y <- c(730,  910, 840, 150, 270, 580, 490, 530, 200, 420, 1100, 1100, 610, 670, 750)
d <- x-y
qqnorm(d); qqline(d)
shapiro.test(d)

        Shapiro-Wilk normality test
data:  d
W = 0.923, p-value = 0.2137
     # We do NOT reject H0 at 0.05. Its considered coming from a normal distribution.
t.test(d, mu=0)

        One Sample t-test
data:  d
t = 6.0234, df = 14, p-value = 3.126e-05
     # We reject H0 at 0.05. The true men of the differences is significantly different from 0.
     # That is, Garage I is charging significantly more than Garage II.
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 39.49412 83.17254
sample estimates:
mean of x
 61.33333
```

Test on a single population variance $\left(\sigma^2\right)$:

$$H_0 : \sigma^2 = \sigma_0^2 \qquad vs. \qquad \begin{aligned} H_1 &: \sigma^2 > \sigma_0^2 \\ H_1 &: \sigma^2 < \sigma_0^2 \\ H_1 &: \sigma^2 \neq \sigma_0^2 \end{aligned}$$

$$\text{T.S. } \chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \quad \sim \chi^2_{df=(n-1)}$$

Test on two population variances $\left(\sigma_1^2, \sigma_2^2\right)$:

$$H_0 : \sigma_X^2 = \sigma_Y^2 \qquad vs. \qquad \begin{aligned} H_1 &: \sigma_X^2 > \sigma_Y^2 \\ H_1 &: \sigma_X^2 < \sigma_Y^2 \\ H_1 &: \sigma_X^2 \neq \sigma_Y^2 \end{aligned}$$

$$\text{T.S. } F = \frac{s_X^2}{s_Y^2} \quad \sim \quad F_{(n_X-1,\, n_Y-1)}$$

Test on a single population proportion (p)**:**

$$H_0 : p = p_0 \qquad vs. \qquad \begin{aligned} &H_1 : p > p_0 \\ &H_1 : p < p_0 \\ &H_1 : p \neq p_0 \end{aligned}$$

$$\text{T.S. } Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \quad \sim N(0,1)$$

Test on two population proportions $\left(\sigma_1^2, \sigma_2^2\right)$**:**

$$H_0 : p_1 = p_2 \qquad vs. \qquad \begin{aligned} &H_1 : p_1 > p_2 \\ &H_1 : p_1 < p_2 \\ &H_1 : p_1 \neq p_2 \end{aligned}$$

$$\text{T.S. } Z = \frac{\hat{p_1} - \hat{p_2}}{\sqrt{\hat{p}\,(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad \sim N(0,1),$$

$$\text{where } \hat{p} = \frac{n_1 Y_1 + n_2 Y_2}{n_1 + n_2} = \text{(pooled proportion)}.$$