

# Chapter 1. Probability

## Objectives

- Basic Concepts
- Enumeration
- Conditional Probability
- Independent Events
- Bayes' Theorem

## Probability

**Ex 1.** If  $P(A) = 0.4$ ,  $P(B) = 0.5$ , and  $P(A \cap B) = 0.3$ , find (a)  $P(A \cup B)$ , (b)  $P(A \cap B')$ , and (c)  $P(A' \cup B')$ .

**Definition 1.** *Probability is a real-valued function  $P(\cdot)$  that assigns each event  $A \subset O$ , the sample space, a number  $P(A)$ , the probability of the event  $A$ , such that:*

1.  $P(A) \geq 0$ ,
2.  $P(O) = 1$ ,
3. If  $A_1, A_2, \dots$  are events and  $A_i \cap A_j = \emptyset$ ,  $i \neq j$ , then

$$P(A_1 \cup A_2 \cup \dots \cup A_k) = P(A_1) + P(A_2) + \dots + P(A_k)$$

*for each positive integer  $k$ , and*

$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$$

*for an infinite, but countable number of events.*

**Theorem 1.** *For each event  $A$ ,*

$$P(A) = 1 - P(A')$$

*Proof.*

$$O = A \cup A' \quad \text{and} \quad A \cap A' = \emptyset$$

$$1 = P(A) + P(A') \quad (\text{why?})$$

$$\therefore P(A) = 1 - P(A')$$

□

**Theorem 2.**

$$P(\emptyset) = 0$$

*Proof.*

□

**Theorem 3.** *If events  $A$  and  $B$  are such that  $A \subset B$ , then  $P(A) \leq P(B)$ .*

*Proof.* Write  $B = A \cup (B \cap A')$  and note that  $A \cap (B \cap A') = \emptyset$ .

$$\text{Therefore, } P(B) = P(A) + P(B \cap A') \geq P(A). \quad (\text{Why?})$$

□

**Theorem 4.** *For any event  $A$ ,  $P(A) \leq 1$ .*

*Proof.*

□

**Theorem 5.** *If  $A$  and  $B$  are any two events, then  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .*

*Proof.* Write  $A \cup B = A \cup (A' \cap B)$  and  $B = (A \cap B) \cup (A' \cap B)$ .

This means

$$\begin{aligned} P(A \cup B) &= P(A) + P(A' \cap B) & (\text{Why?}) \\ &= P(A) + \{P(B) - P(A \cap B)\} \end{aligned}$$

□

**Theorem 6.** *If  $A$ ,  $B$ , and  $C$  are three events, then*

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

*Proof.*

□

**Definition 2.** *Methods of enumeration:*

1. *There is a total of  $n!$  ways of arranging  $n$  distinct objects (i.e.,  $n!$  ways of permutation).*
2. *There is a total of  ${}_nP_r$  ways of selecting  $r$  and arranging them out of  $n$  distinct objects.*
3. *There is a total of  ${}_nC_r$  ways of selecting  $r$  out of  $n$  distinct objects.*

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

$${}_nP_r = \frac{n!}{(n-r)!} = n(n-1) \cdots (n-r+1)$$

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

**Ex 2.** Find the total number of ways of arranging for distinct letters, like  $a, b, c$  and  $d$ .

**Ex 3.** Find the total number of ways of arranging  $p, e, p, p, e, r$ .

**Ex 4.** Find the total number of ways of selecting four “officials” from 4 candidates.

**Ex 5.** Find the total number of ways of selecting a president, a vice president, a secretary, and a treasurer from 4 candidates.

**Ex 6.** Find the total number of all possible four-letter code words from the 26 letters in the alphabet.

**Ex 7.**

1. Find the total number of all possible five-card hands drawn from a regular deck of 52 playing cards.
2. Find the total number of ways of selecting three kings and two queens in a five-card hand.
3. Find the “probability” of selecting three kings and two queens in a five-card hand.

**Ex 8.** Suppose on a boat there are seven colored flags; four oranges and three blue flags. Find the total number of ways of arranging seven flags on a vertical pole.

**Ex 9.** Show the following and explain the meaning in *plain* terms.

$${}_nP_r = \binom{n}{r} \cdot r!$$

**Ex 10.** Show

$$(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r.$$

**Definition 3.** *Conditional probability of an event  $A$  given that event  $B$  has occurred, is*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

*provided that  $P(B) > 0$ .*

**Ex 11.** The following table classifies 1,456 people by their gender and by whether or not they favor a gun law.

|                  | Male( $B_1$ ) | Female ( $B_2$ ) | Totals |
|------------------|---------------|------------------|--------|
| Favor ( $A_1$ )  | 392           | 649              | 1,041  |
| Oppose ( $A_2$ ) | 241           | 174              | 415    |
| Totals           | 633           | 823              | 1,456  |

Find the following probabilities if one of these 1,456 people is selected randomly.

1.  $P(A_1)$
2.  $P(A_1|B_1)$
3.  $P(A_1|B_2)$
4. Interpret your answers in “plain” terms.

**Ex 12.** If  $P(A) = 0.4$ ,  $P(B) = 0.5$ , and  $P(A \cap B) = 0.3$ , find  $P(A|B)$  and  $P(B|A)$ .

**Theorem 7.** *If  $P(B) > 0$ , then*

1.  $P(A|B) \geq 0$ ,
2.  $P(B|B) = 1$ ,
3. *If  $A_1, A_2, A_3, \dots$  are mutually exclusive, then  $P(A_1 \cup \dots \cup A_k|B) = P(A_1|B) + \dots + P(A_k|B)$*

*Proof.*

□

**Definition 4.** *Multiplication rule:*

$$P(A \cap B) = P(A)P(B | A) \\ \text{or } P(A \cap B) = P(B)P(A | B)$$

**Ex 13.** From a deck of playing cards, find the probability that the third spade appears on the sixth draw.

*Answer:* Let  $A$  = Two spades are drawn in the first five cards;  $B$  = a spade on the sixth draw. Then,

$$P(A) = \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} = 0.274, \quad \text{and} \quad P(B | A) = \frac{11}{47} = 0.234 \\ \therefore P(A \cap B) = P(A)P(B | A) = (0.274)(0.234) = 0.064$$

**Ex 14.** An insurance company sells a number of policies; among these are 60% for autos, 40% for homeowners, and 20% are for both of these two. Let  $A_1$  = people with only an auto policy,  $A_2$  = with only homeowners,  $A_3$  = with both, and  $A_4$  = with only other types of policies. If a person is randomly selected from the policy holders, then  $P(A_1) = 0.4$ ,  $P(A_2) = 0.2$ ,  $P(A_3) = 0.2$ , and  $P(A_4) = 0.2$ , as these four events are mutually exclusive and exhaustive. Further, let  $B$  be the event that a policy holder will renew at least one of the auto or homeowner's policies. Also, from past experience, we have  $P(B | A_1) = 0.6$ ,  $P(B | A_2) = 0.7$ , and  $P(B | A_3) = 0.8$ . Find the conditional probability that a person selected at random will renew at least one of these policies.

*Answer:*

$$P(B | A_1 \cup A_2 \cup A_3) = \frac{P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)}{P(A_1) + P(A_2) + P(A_3)} \\ = \frac{(0.4)(0.6) + (0.2)(0.7) + (0.2)(0.8)}{0.4 + 0.2 + 0.2} \\ = \frac{0.54}{0.80} = 0.675$$

**Ex 15.** A device has two components,  $C_1$  and  $C_2$ . The probability that each will fail when both are in operation is 0.01 for a one-year period. However, when one fails, the probability of the other failing is 0.03 due to added strain. Find the probability that the device fails in one year.

*Answer:*

$$P(C_1 \text{ fails})P(C_2 \text{ fails} | C_1 \text{ fails}) + P(C_2 \text{ fails})P(C_1 \text{ fails} | C_2 \text{ fails}) \\ = (0.01)(0.03) + (0.01)(0.03) = 0.0006$$

**Definition 5.** *Events  $A$  and  $B$  are **independent** events if and only if  $P(A \cap B) = P(A)P(B)$ .*

Notice that this implies  $P(A | B) = P(A)$  and  $P(B | A) = P(B)$  when  $A$  and  $B$  are independent.

**Theorem 8.** If  $A$  and  $B$  are independent, then the following pairs of events are also independent:

1.  $A$  and  $B'$ ,
2.  $A'$  and  $B$ ,
3.  $A'$  and  $B'$

*Proof.*

$$\begin{aligned} P(A \cap B') &= P(A)P(B' | A) = P(A)\{1 - P(B | A)\} \\ &= P(A)\{1 - P(B)\} \quad (\text{Why?}) \\ &= P(A)P(B') \end{aligned}$$

$$\begin{aligned} P(A' \cap B') &= P\{(A \cup B)'\} = 1 - P(A \cup B) \\ &= 1 - \{P(A) + P(B) - P(A \cap B)\} \\ &= 1 - \{P(A) + P(B) - P(A)P(B)\} \quad (\text{Why?}) \\ &= \{1 - P(A)\} - P(B)\{1 - P(A)\} \\ &= \{1 - P(A)\}\{1 - P(B)\} \\ &= P(A')P(B') \end{aligned}$$

□

**Ex 16.** A red and a white die are rolled. Let  $C = \{5 \text{ on red die}\}$  and  $D = \{\text{sum of two dice is } 11\}$ . Check if  $C$  and  $D$  are independent.

*Answer:* Note that  $P(C) = \frac{1}{6}$  and  $P(D) = \frac{2}{36} = \frac{1}{18}$ . (Why?)

Also,  $P(C \cap D) = \frac{1}{36}$ . That is,  $P(C)P(D) \neq P(C \cap D)$ .

$\therefore C$  and  $D$  are \_\_\_\_\_.

**Ex 17.** An urn contains for balls numbered 1, 2, 3, and 4. One ball is drawn at random. Let the events be defined as  $A = \{1, 2\}$ ,  $B = \{1, 3\}$ ,  $C = \{1, 4\}$ . Using this example, show that “pair-wise” independence does not always imply “complete” independence.

*Answer:*

$$\begin{aligned} P(A \cap B) &= \frac{1}{4} = P(A)P(B) \\ P(A \cap C) &= \frac{1}{4} = P(A)P(C) \\ P(B \cap C) &= \frac{1}{4} = P(B)P(C) \end{aligned}$$

$$\text{But, } P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C).$$

**Ex 18.** A rocket has a built-in redundant system. If component  $K_1$  fails, it is bypassed and component  $K_2$  is used. If component  $K_2$  fails, it is bypassed and component  $K_3$  is used. Suppose that the probability of failure of any one of these components is 0.15 and assume that the failures

of these components are mutually independent events. Let  $A_i$  be the event that component  $K_i$  fails for  $i = 1, 2, 3$ . Find the probability that the system does not fail.

*Answer:*

$$\begin{aligned} P\{(A_1 \cap A_2 \cap A_3)'\} &= 1 - P(A_1 \cap A_2 \cap A_3) \\ &= 1 - P(A_1)P(A_2)P(A_3) \\ &= 1 - (0.15)^3 \\ &= 0.9966 \end{aligned}$$

**Definition 6.** Events  $A$ ,  $B$  and  $C$  are **mutually independent** if and only if the following two conditions hold:

1. They are pairwise independent, i.e.,  $P(A \cap B) = P(A)P(B)$ ,  $P(A \cap C) = P(A)P(C)$ , and  $P(B \cap C) = P(B)P(C)$ .
2.  $P(A \cap B \cap C) = P(A)P(B)P(C)$

**Theorem 9.** If  $A$ ,  $B$ , and  $C$  are mutually independent events, then the following events are also independent:

1.  $A$  and  $(B \cap C)$ ,
2.  $A$  and  $(B \cup C)$ ,
3.  $A'$  and  $(B \cap C')$ .

*Proof.*

$$\begin{aligned} P\{A \cap (B \cap C)\} &= P(A \cap B \cap C) \\ &= P(A)P(B)P(C) \quad (\text{Why?}) \\ &= P(A)P\{(B \cap C)\} \\ P\{A \cap (B \cup C)\} &= \underline{\hspace{4cm}} \\ &= \underline{\hspace{4cm}} \\ &= \underline{\hspace{4cm}} \\ P\{A' \cap (B \cap C')\} &= P(A' \cap B \cap C') \\ &= P(B)\{P(A' \cap C')|B\} \quad (\text{Why?}) \\ &= P(B)\{1 - P(A \cup C)|B\} \\ &= P(B)\{1 - P(A \cup C)\} \quad (\text{Why?}) \\ &= P(B)P(A')P(C') \quad (\text{Why?}) \\ &= P(A')P(B \cap C') \quad (\text{Why?}) \end{aligned}$$

□

**Ex 19.** A fair six-sided die is rolled six times. Let  $A_i$  = the event that side  $i$  is rolled on the  $i$ th roll, called a “match” on the  $i$ th trial,  $i = 1, 2, \dots, 6$ . Thus,  $P(A_i) = 1/6$  and  $P(A') = 5/6$ . Let  $B$  = the event that at least one match occurs. Find  $P(B)$ .

*Answer:* Let  $B'$  = the event that no matches occur.

$$\begin{aligned} P(B) &= 1 - P(B') = 1 - P(A'_1 \cap A'_2 \cap \dots \cap A'_6) \\ &= 1 - \left(\frac{5}{6}\right)^6 \\ &= 0.6651 \end{aligned}$$

**Theorem 10. Bayes' Theorem**

Let  $O = B_1 \cup B_2 \cup \dots \cup B_m$  and  $B_i \cap B_j = \emptyset, i \neq j$ , (i.e.,  $B_i$ 's are mutually exclusive and exhaustive partition of the sample space  $O$ ), then

$$P(B_k|A) = \frac{P(B_k)P(A|B_k)}{\sum_{i=1}^m P(B_i)P(A|B_i)}, \quad k = 1, 2, \dots, m.$$

It's often summarized by this statement:

$$(\text{posterior probability}) \propto (\text{prior probability}) \times (\text{likelihood})$$

*Proof.*

□

**Ex 20.** Machines I, II, and III are all producing springs of the same length. Of their production, machines I, II, and III produce 2%, 1%, and 3% defective springs, respectively. Of the total production of springs in the factory, machine I produces 35%, machine II produces 25%, and machine III produces 40%. (a) If one spring is selected at random from the total produced in a day, what is the probability that it's defective? (b) If the selected spring is defective, what is the conditional probability that it was produced by machine III?

*Answer:*

$$\begin{aligned} P(D) &= P(I)P(D|I) + P(II)P(D|II) + P(III)P(D|III) \\ &= (0.35)(0.02) + (0.25)(0.01) + (0.4)(0.03) = 0.0215 \end{aligned}$$

$$P(III|D) = \frac{P(III)P(D|III)}{P(D)} = \frac{(0.4)(0.03)}{0.0215} = 0.5581$$



**Ex 21.** Let  $T^+$  = test positive,  $T^-$  = test negative;  $C^+$  = has cancer, and  $C^-$  = does not have cancer. A pap smear test is used to detect cervical cancer and is known to show about 16% *false negatives*, and about 19% *false positives*, i.e.,  $P(T^-|C^+) = 0.16$  and  $P(T^+|C^-) = 0.19$ . In the US, there are about 8 women in 100,000 with this cancer. Find the probability that a randomly selected woman who was just tested positive actually has the cancer.

*Answer:*

$$\begin{aligned}
 P(C^+|T^+) &= \frac{P(C^+ \cap T^+)}{P(T^+)} \\
 &= \frac{P(T^+|C^+)P(C^+)}{P(T^+|C^+)P(C^+) + P(T^+|C^-)P(C^-)} \\
 &= \frac{(0.84)(0.00008)}{(0.84)(0.00008) + (0.19)(0.99992)} \\
 &= 0.000354
 \end{aligned}$$

**Ex 22.** There is a new diagnostic test for a disease that occurs in about 0.05% of the population. The test is not perfect but will detect a person with the disease 99% of the time. It will, however, say that a person without the disease has the disease about 3% of the time. A person is selected at random from the population and the test indicates that this person has the disease. What are the conditional probabilities that (a) the person has the disease? (b) the person does not have the disease?

*Answer:* As you can tell, use of proper notations makes a big difference to handle problems like this. We have

$$P(D) = 0.0005, P(+|D) = 0.99, \text{ and } P(+|\bar{D}) = 0.03.$$

$$P(D|+) = \underline{\hspace{10cm}}$$

$$= \underline{\hspace{10cm}}$$

$$P(\bar{D}|+) = \underline{\hspace{10cm}}$$

$$= \underline{\hspace{10cm}}$$