“Circular Reasoning”

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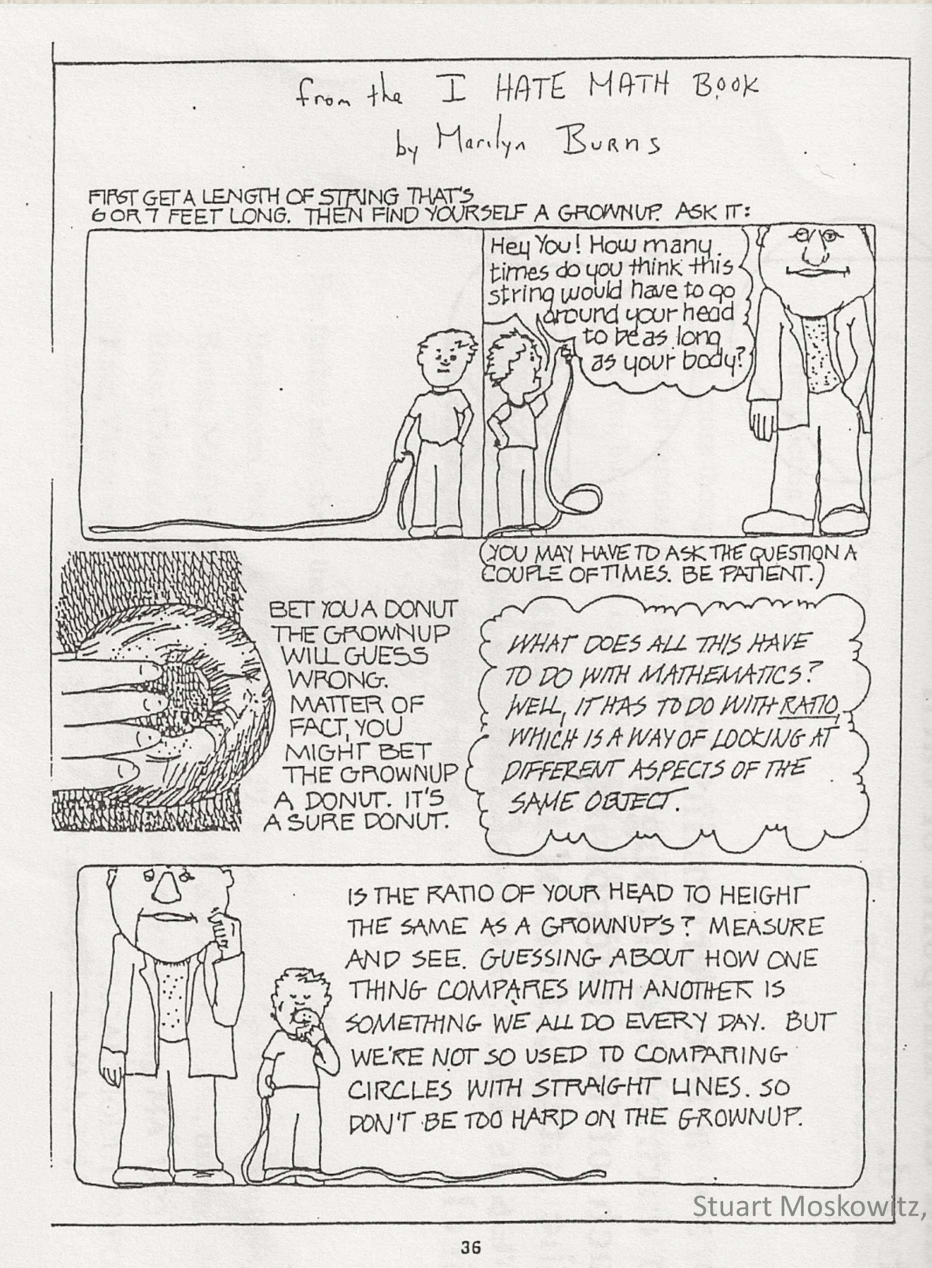
π

It is one of the great mysteries why nature seems to know mathematics.

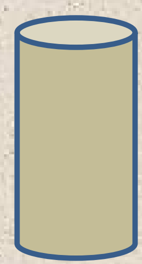
Richard Preston

The Mountains of PI, New Yorker Magazine, March 2, 1992

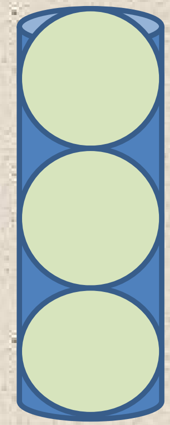
We are so linear. We are so not used to thinking in circles.



Which measurement is longer, the length or the circumference of a toilet paper tube?



Which measurement is longer, the length or the circumference of a tennis ball can?



- The square is a very easy figure to calculate with. If you want to know the distance around its outer perimeter, all you have to do is measure one side and multiply by four; if you want to know its area, you simply need to measure one side and multiply the measurement by itself—that is, square it.
- The circle, by contrast, is a very difficult figure indeed to calculate with, … Since π can never be expressed precisely by numbers, you can know exactly either the diameter or the circumference of a circle, but never both.

  - *Techniques for Geometric Transformation*
    - *John Michael Greer*

Estimating PI (*not PIE!!*)

- Measure with a measuring tape (or with a string and a ruler) the circumference and diameter of various circular objects (e.g. jar lids, pan lids, plates, clocks). Record the circumference and diameter of each object in the table below, then compute *C/D* for each entry.

OBJECT	CIRCUMFERENCE	DIAMETER	C/D

Long ago…..

- Just as we discovered with results on previous page, early people knew the ratio of the circumference to the diameter was constant.
- They knew it was a little more than 3 to 1.
- They came up with creative ways to describe it, but they were all approximations.

- Here's 3 different values we use to approximate PI today:

3.14   3.1415926535 $3\frac{1}{7}$

What were some of their approximations?

How did they get them?

PI (*by describing circumference*)

1. **The following description appears in the Chinese mathematical manual, *Nine Chapters on the Mathematical Art, originally composed about 200 BCE (Katz, 20). "One has a round field; the circumference is 30 steps, the diameter 10 steps."*** **What value is being used for π? Explain.**

2. **The following description appears in the Hebrew *Bible (Old Testament of the Christian Bible) in I Kings 7:23. The passage concerns events that occurred during the reign of King Solomon in about 950 BCE (Katz, 20). "And he made a molten sea of ten cubits from brim to brim, round in compass . . . and a line of thirty cubits did compass it round about." [A "molten sea" is a large circular vessel.]***

   What value is being used for π? Explain.

3. **The following description appears in the *Aryabhatiya, written by the Indian mathematician Aryabhata, who was born in 476 CE (Berggren, et al, Appendix 1, 679). "Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000."*** **What value is being used for π? Explain.**

Ahmes (Rhind) Papyrus Problem 50 (uses Area to find π)

- Example of a round field of a diameter 9 khet. What is its area?

- Take away 1/9 of the diameter, namely 1; the remainder is 8.
- Multiply 8 times 8; it makes 64.

- Therefore it contains 64 setat of land.

How'd the Egyptians Get Their Area Formula?

From Paulus Gerdes

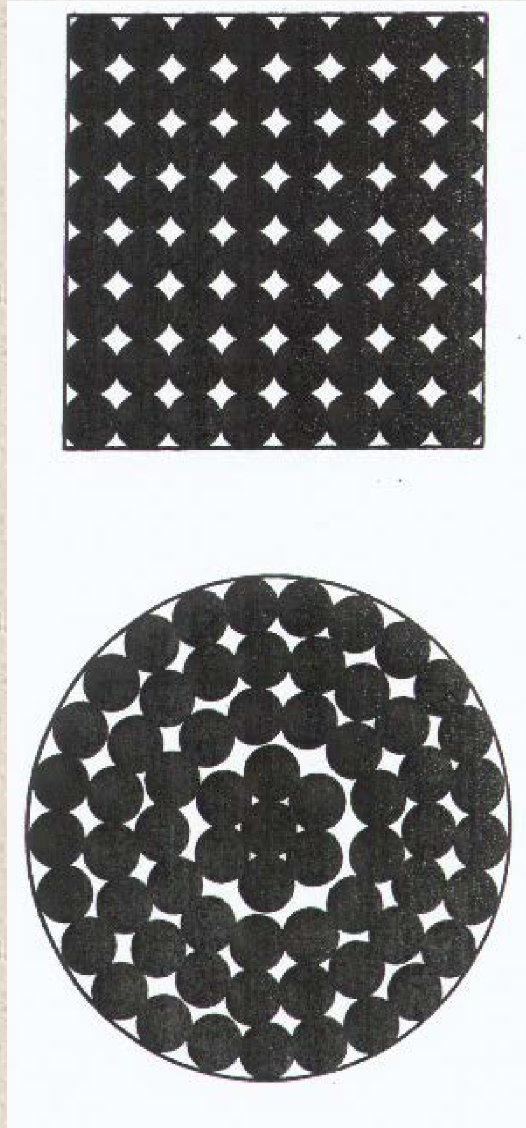
- 64 tokens form an 8 by 8 grid with an area of …. 64 tokens.

- Rearrange these 64 tokens to form a circle.
- What's the circle area?
- Observe the diameter.
- It's 9 tokens.

- So if diameter (d) = 9 units, then the side of the square (s) = 8 units,

$$s = \frac{8}{9}D \quad so \quad A = s^2 = \left(\frac{8}{9}D\right)^2$$

What value did the Egyptians use for PI?



Finding Egyptian Value for π

They used $A = \left(\frac{8}{9}D\right)^2$ and we use $A = \pi r^2$

$$\left(\frac{8}{9} \times 2r\right)^2 = \pi r^2$$

$$\left(\frac{16}{9}r\right)^2 = \pi r^2$$

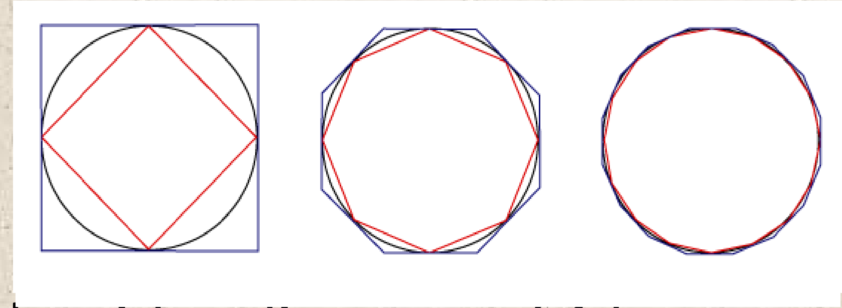
$$\left(\frac{16}{9}\right)^2 r^2 = \pi r^2$$

$$\frac{256}{81}r^2 = \pi r^2$$

$$\pi = \frac{256}{81}$$

$$\pi \approx 3.1605$$

Archimedes (287-212 BCE)



- Construct a circle with a diameter of 1 between two similar polygons (one inscribing and the other circumscribing the circle).
- The circumference of the circle is between the perimeters of the 2 polygons.
- Repeatedly increase the number of sides of the polygons and recalculate until the circle is “exhausted”, eventually a polygon will be reached whose sides are so short that it will coincide with the circle.
- Archimedes concluded after calculating the 96-gon that :

$$3\frac{10}{71} < \pi < 3\frac{10}{70}$$

What value did Archimedes get for PI?

Archimedes and His Method of Exhaustion

- http://www.math.utah.edu/~alfeld/Archimedes/Archimedes.html
- http://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html

for excellent notes on Archimedes and his derivation of PI

- http://www.pbs.org/wgbh/nova/archimedes/palimpsest.html .
- This website has an excellent link showing a visual on Archimedes Method, as well as a link to details about the Palimpsest

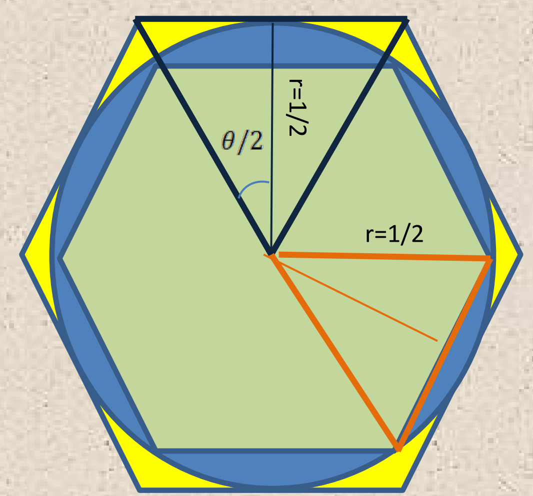
For a unit circle with *diameter = 1 and radius = 1/2*

n = number of sides of polygon

θ = 360 degrees/*n*

$$P_{inscribed\ polygon} = n \times \sin(\theta/2)$$

$$P_{circumscribed\ polygon} = n \times \tan(\theta/2)$$



Of course, there was no algebra and trigonometry. Instead, Archimedes did it with ratios of similar triangles!!

How Do We Know About Archimedes?

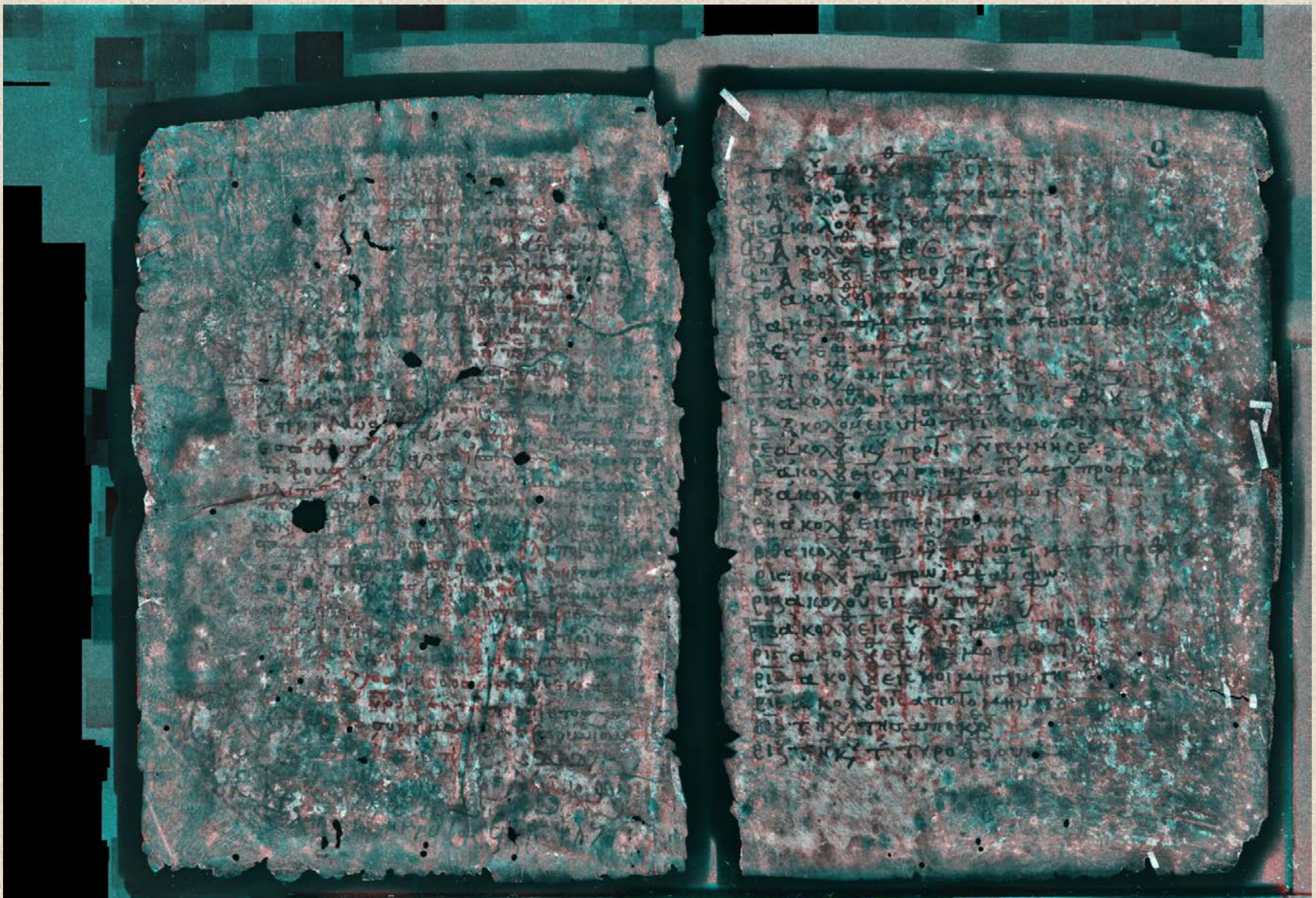
THE PALIMPSEST:

--A recycled ancient manuscript.

--The original text is scratched off.

--New text is written on top of the old

http://archimedespalimpsest.org/about/history/

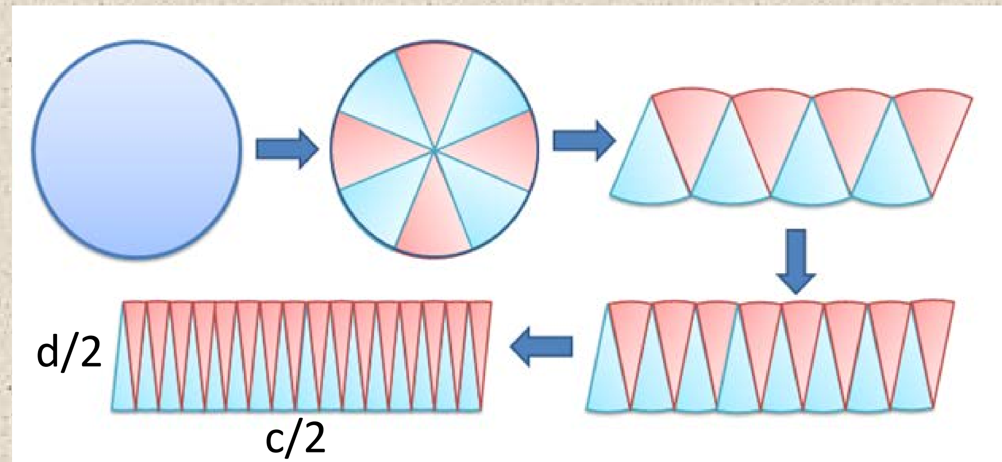


Download all the original high resolution scans of the pages of Archimedes Palimpsest by clicking on "Image Bank" at http://www.archimedespalimpsest.org/

OLD CHINESE METHOD FOR FINDING AREA OF A CIRCLE USING PARALLELOGRAMS

given the circumference and diameter

$$A = \frac{C}{2} \times \frac{D}{2}$$

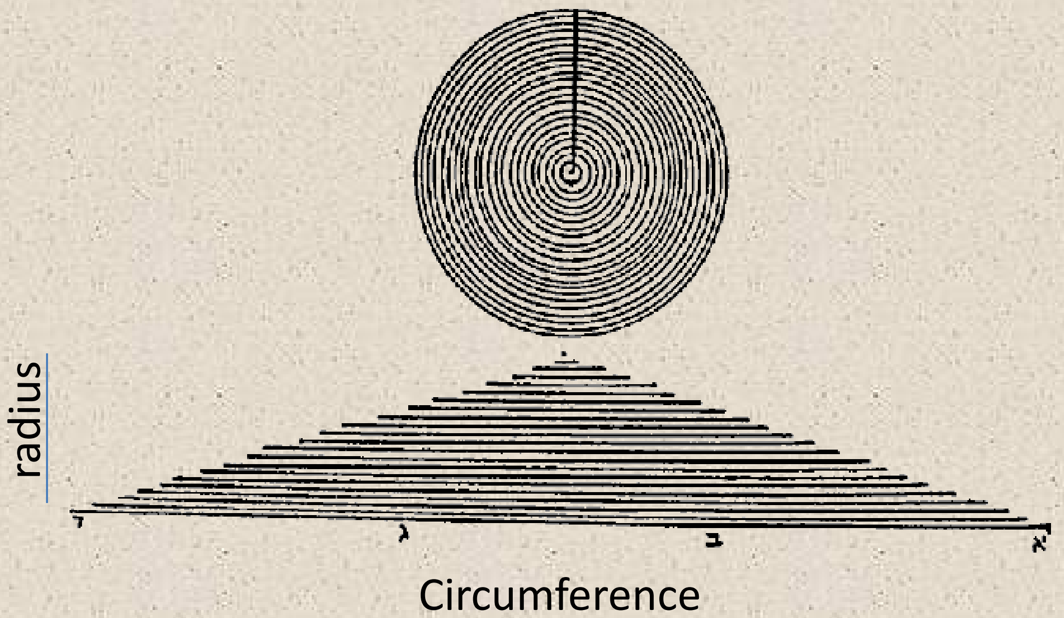


$$A = \frac{2\pi r \times 2r}{4} = \frac{4\pi r^2}{4} = \pi r^2$$

OLD SPANISH METHOD FOR FINDING AREA OF A CIRCLE USING TRIANGLES

Drawn by Rabbi Abraham bar Hiyya Hanasi (c 1100 CE)

Sliced concentric rings are straightened and reformed into a triangle.



$$A = \frac{1}{2}BH = \frac{1}{2}Cr = \frac{1}{2}(2\pi r)r = \pi r^2$$

Ludolf Van Ceulen spent much of his life (1540-1610) calculating PI to 35 digits using Archimedes method. He used polygons with 2^{62} (that's 32 billion) sides!!

He then had it engraved on his tombstone.

Tombstone. *The tombstone of Ludolph van Ceulen in Leiden, the Netherlands, is engraved with his amazing 35-digit approximation to pi. Notice that, in keeping with the tradition started by Archimedes, the upper and lower limits are given as fractions rather than decimals. (Photo courtesy of Karen Aardal.*

The Renaissance introduced new Algebraic strategies. The goal with Infinite Series to Find PI is to find Series that converge quickly.

http://www.geom.uiuc.edu/~huberty/math5337/groupe/expresspi.html

Leibniz's (1674) was easy to understand but very slow to generate PI (300 terms to get 2 digits of accuracy):

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

Euler's (1748) was slow, too: $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{k^2}$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \quad or \quad \pi = \sqrt{6\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right)}$$

Computations of Pi
Pi Timeline through 2000

http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Pi_chronology.html

- Chudnovskys
  - First to reach 1 billion digits
- Kanada
  - 206 billion digits as of 1999

- Every digit has an equal chance of appearing in pi – can't prove

Chudnovsky Brothers

- Greg Chudnovsky
  - Number theorist
- David Chudnovsky
  - Older brother
  - Mathematician
  - Lives 5 blocks away from Gregory
- Work at Brooklyn Polytechnic University
- Homemade computer
  - $70,000 from wives' work
  - Can't shut down
  - Requires a lot of air conditioning

Gregory, left, and David Chudnovsky, the two brothers who head up the Institute for Mathematics and Advanced Supercomputers at Polytechnic University in Downtown Brooklyn.

Photo courtesy of Polytechnic University

"THEY WONDER WHETHER THE DIGITS CONTAIN A HIDDEN RULE, CLOSE TO THE MIND OF GOD."

RICHARD PRESTON

Chudnovsky’s eventually calculated PI to over 4 billion decimal digits after the New Yorker story was published.

The record has been broken several times since the timeline was last updated.

BUFFON'S NEEDLES

- Find PI using probability:
- The reciprocal of PI equals the probability of a needle landing on a gridline (if the distance between the gridlines is twice the length of the needle).
- Captain O. C. Fox's claim to fame happened while recovering from a wound sustained during the American Civil War in 1864.   He dropped a needle 1100 times and derived PI to 2 decimal places.

AND IF PROBABILITY ISN'T A STRANGE ENOUGH METHOD FOR FINDING PI:

- Pi also appears as the average ratio of the actual length and the direct distance between source and mouth in a meandering river (Stølum 1996, Singh 1997).

Then there are those who choose to memorize digits of PI

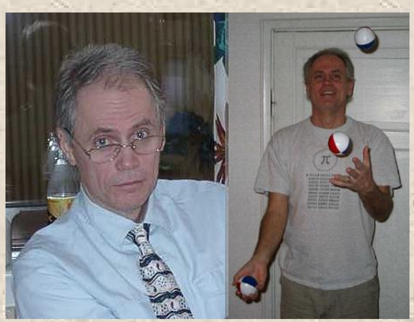
Gaurav Raja, a teenage pi reciter, broke the 27-year-old North American record in 2006 by reciting 10,980 digits!

- http://www.teachpi.org/stories.htm

The current world record, according to The Guinness Book of Records is 67,890 digits, accomplished by **Chao Lu** in China in 2005. It took Him 24 hours and 4 minutes!!

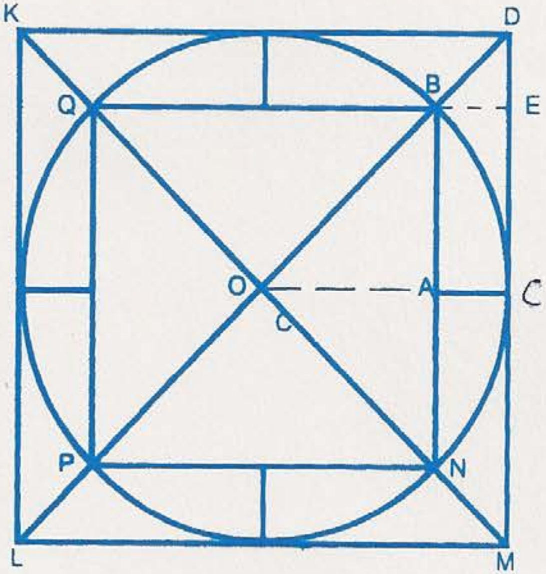
- http://www.guinnessworldrecords.com/world-records/1/most-pi-places-memorised
- http://pi-world-ranking-list.com/lists/memo/index.html

In 2005, **Matt Bergsten** recited 9778 digits in 1 hour 20 minutes 45 seconds while juggling!



The race of circle squarers will never die out as long as ignorance and the thirst for glory remain united.

H Schubert, 1899



"It is through intuition that the mathematical world remains in touch with the real world".

AREA OF A CIRCLE & EXACT Pi

LM = PB = a. The reasoning in finding the areas of ACB and BCD from the dimensions is intuitive rather than deductive and very simple.
Quadrilateral = ACDB

$$\text{Area of ACB} = \left(\frac{\text{Quadrilateral area}}{2}\right)\left(\frac{3\ AC + 2\ AB}{PB}\right)$$

$$\text{Area of BCD} = \left(\frac{\text{Quadrilateral area}}{2}\right)\left(\frac{AC + BD + CD}{LM}\right)$$

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SIVA METHOD FOR THE TOTAL AREA OF A CIRCLE

The conventional formula to calculate the area of a circle is πr^2 where 'π' is a constant representing a ratio of the circumference of a circle to its diameter, and 'r' is the radius of the circle. In this method the entire area of the circle is calculated without the help of the constant π.

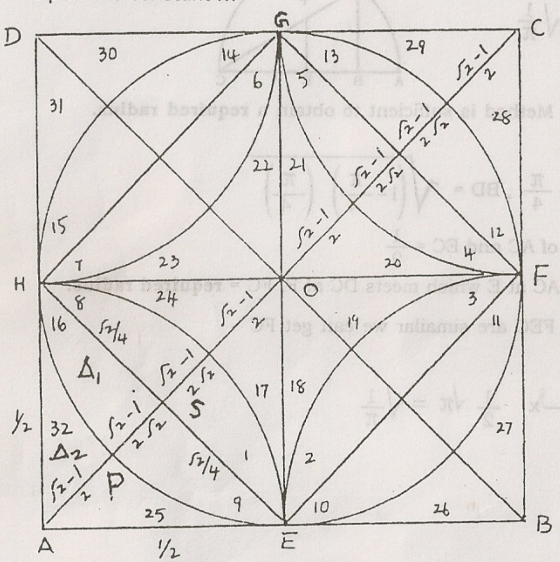


Fig 1

Method

Draw a square ABCD with a side 1. E, F, G and H are the midpoints of four sides. Join AC and BD to get two diagonals. Similarly, join EG and FH which are equal to the side of the ABCD square. AC, BD, EG and FH intersect at O. Take O as the centre and half the side of the square ABCD as radius inscribe a circle with the square. The side of the square and the diameter of the inscribed circle are thus the same.

Join E, F, G and H to get another square EFGH which is smaller than the ABCD square. Take A, B, C and D as the centres and with the same radius draw four arcs as shown in the figure. At the end we find that the ABCD square is divided exactly into 32 parts and are numbered 1 to 32. 1 to 16 are of one dimension called Δ_1 and 17 to 32 are of another dimension called Δ_2.

The third square AEOH is 1/4th of the ABCD square and contains four triangles ASH, ASE, ESO and OSH. Each triangle is divided into two parts. For example the triangle ASH is divided into SPH called Δ_1 and APH called Δ_2.

1. ABCD = a square
2. AB = BC = CD = AD = 1
3. Radius = $r = \frac{1}{2}$
4. AC = $\sqrt{1+1} = \sqrt{2}$
5. OA = $\frac{\sqrt{2}}{2}$ = HE
6. HS = $\frac{HE}{2} = \frac{\sqrt{2}}{4}$
7. AS = $\frac{AO}{2} = \frac{\sqrt{2}}{4}$
8. AP = AO - radius = $\frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{\sqrt{2}-1}{2}$
9. SP = OP - OS = $\frac{1}{2} - \frac{\sqrt{2}}{4} = \frac{2-\sqrt{2}}{4}$

Area of the triangle ASH is denoted by Δ and has SPH = Δ_1 and APH = Δ_2. Each one has three sides - in SPH : SP, SH and a common side an arc HP; in APH : AP, AH and the arc HP.

10. Area of the triangle ASH = $\Delta = \frac{1}{16}$

To deduce the following two formulae, π equal to 3.1415926....... and $(14 - \sqrt{2})/4$ of **Gayatri method** have been considered. The latter algebraic π number has ultimately established itself a right choice.

11. Area of $\Delta_1 = \frac{\Delta}{2}\left[SH + SP\,(4+\sqrt{2})\right] = \frac{\frac{1}{16}}{2}\left[\frac{\sqrt{2}}{4} + \left(\frac{2-\sqrt{2}}{4}\right)(4+\sqrt{2})\right]$

$= \frac{6-\sqrt{2}}{128}$ - 1

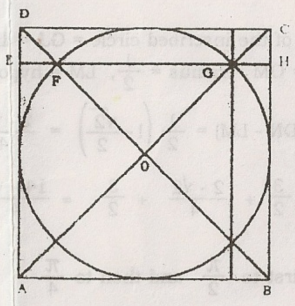
12. Area of $\Delta_2 = \frac{\Delta}{2}\left[AH + AP\left(\frac{2+\sqrt{2}}{2}\right)\right] = \frac{\frac{1}{16}}{2}\left[\frac{1}{2} + \left(\frac{\sqrt{2}-1}{2}\right)\left(\frac{2+\sqrt{2}}{2}\right)\right]$

$= \frac{2+\sqrt{2}}{128}$ - 2

13. The sum of the areas of Δ_1 and Δ_2 is exactly equal to the area of the triangle ASH = $\Delta = \frac{1}{16}$
14. In the square ABCD, the inscribed circle has 16 Δ_1 and 8 Δ_2.
15. $16\,\Delta_1 + 8\,\Delta_2 = \pi r^2$ (where $r = \frac{1}{2}$) $= \frac{\pi}{4} \Rightarrow 16\left(\frac{6-\sqrt{2}}{128}\right) + 8\left(\frac{2+\sqrt{2}}{128}\right) = \frac{\pi}{4}$

$\therefore\ \pi = \frac{14-\sqrt{2}}{4}$

GAYATRI METHOD FOR THE CIRCUMFERENCE OF A CIRCLE
(STARTING OR HYPOTHETICAL METHOD)



AB = 1 = Side of the square = diameter of the inscribed circle = d = 1

OF = $\frac{1}{2}$ = radius

FOG = right angled triangle

EH = 1

FG = Hypotenuse of FOG triangle = $\frac{\sqrt{2}}{2}$

GH = $\frac{EH - FG}{2} = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$

Circumference of the inscribed circle =

$$AB + AD + DC + CH = 1 + 1 + 1 + \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right) = \mathbf{\frac{14-\sqrt{2}}{4}}$$

Source: "**A proof for the exact value of π**" (2001, 84 Pages) by **R. Sarva Jagannadha Reddy**, Lecturer in Zoology, PVKN Govt. College, **Chittoor - 517 002**. A.P. India.

1897 -- INDIANA HOUSE BILL No. 246

- “A Bill for an act introducing a new mathematical truth (the squaring of the circle) and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the Legislature of 1897. ….”
- “…… the fourth important fact, that the ratio of the diameter and circumference is as five-fourths to four…..” (the bill actually contains at least 3 different values for PI)

If $\frac{d}{c} = \frac{5/4}{4}$, then $\pi = \frac{c}{d} = \frac{4}{5/4} = \frac{16}{5} = 3.2$

- The bill was referred to the House Committee on Swamp Lands, which passed it on to the Committee on Education, which recommended that it should be passed.

- On February 5, 1897, the House passed the bill 67 to 0 and the bill moved to the Senate.

- After being approved by the Committee on Temperance, a Mathematics professor from Princeton, happened to hear a speech supporting the bill. After coaching the Senate, the bill was tabled and has not been re-introduced.

- The author of the bill, Dr. Edwin Goodwin, MD., also claimed to have trisected the angle and doubled the cube.

- Beckmann, History of PI

Circle Squarers Still Abound!!

- http://mathworld.wolfram.com/PiDigits.html

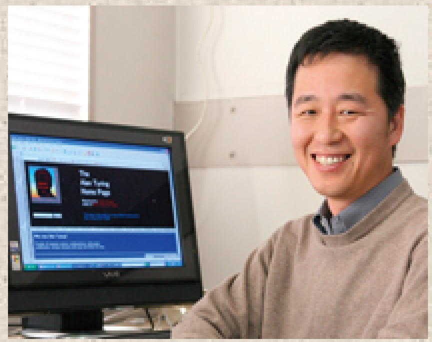
From
http://www.worldrecordsacademy.org/science/most_digits_of_pi_calculated_Shigeru_Kondo_and_Alexander_Yee_sets_world_record_101848.htm

Wednesday, September 1, 2010

Most digits of pi calculated - **Shigeru Kondo** and **Alexander Yee sets world record**

TOKYO, Japan -- **Shigeru Kondo,** a Japanese systems engineer and **Alexander J. Yee**, an American computer science student, have calculated the value of pi to five trillion digits - setting the new world record for the **Most digits of pi calculated.**

*Photo: **Shigeru Kondo**, a 55-year-old resident of Iida and a company employee in Nagano Prefecture, assembled a computer with 32 terabytes of hard-drive capacity and used an application made by Alexander Yee, a 22-year-old student at a US graduate school to calculate the the value of pi to five trillion digits.*



- For a more complete story as well as the specifications of the computer built by Mr. Kondo, go to
- http://gizmodo.com/#!5606273/this-computer-just-calculated-pi-to-a-world-record-5-trillion-digits

Processor
2 x Intel Xeon X5680 @ 3.33 GHz - (12 physical cores, 24 hyperthreaded)
Memory
96 GB DDR3 @ 1066 MHz - (12 x 8 GB - 6 channels) - Samsung (M393B1K70BH1)
Motherboard
Asus Z8PE-D12
Hard Drives
1 TB SATA II (Boot drive) - Hitachi (HDS721010CLA332)
3 x 2 TB SATA II (Store Pi Output) - Seagate (ST32000542AS)
16 x 2 TB SATA II (Computation) - Seagate (ST32000641AS)
Raid Controller
2 x LSI MegaRaid SAS 9260-8i
Operating System
Windows Server 2008 R2 Enterprise x64
Built By
Shigeru Kondo

October 17, 2011:

[The record has been improved to 10 trillion digits.]

Using the Chudnovsky's formula, it takes 371 days of computing.

December 28, 2013:

The record has been improved to 12 trillion digits.

Using the same Chudnovsky Formula, it takes 75 days of computing.

This beats Moore's Law, which states that processing speeds double about every 2 years.

Processor:	2 x Intel Xeon E5-2690 @ 2.9 GHz - (16 physical cores, 32 hyperthreaded)
Memory:	128 GB DDR3 @ 1600 MHz - 8 x 16 GB - 8 channels)
Motherboard:	Asus Z9PE-D8
Hard Drives:	Boot: 1 TB Pi Output: 4 x 3 TB Computation: 24 x 3 TB Backup: 4 x 3 TB
Operating System:	Windows Server 2012 x64

Since I like puzzles so much,   let's close with

A PI PUZZLE

Casey the Rope Maker decided to make a rope long enough to stretch around the Earth (the circumference of which is 25,000 miles). Having completed the task, she discovered she actually had made the rope 12 yards too long. Not wanting to cut the rope (it was somewhat of a record length), Casey decided to put the ends of the rope together and have all her relatives help hold the rope an equal distance off the ground all the way around the earth. Assume they were all able to stand on dry ground. Which of these animals would have enough room to pass easily under the rope on dry ground: (a) microbe, (b)ant, (c) snake, (d) Casey, (e) elephant, (f) blue whale, or (g) none of the above?