Why certain integrals are “impossible”.

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Outline

1. Introduction.

2. Elementary fields and functions.

3. Liouville’s Theorem.

4. An example.
Probability

- Central Limit Theorem

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du \]

- For probability applications, we need \( \Phi(\infty) = 1 \).

- This is not proved by finding a formula for \( \Phi(x) \) (by finding an explicit antiderivative of \( e^{-u^2/2} \)) and taking the limit as \( x \to \infty \).
Number Theory

- Prime Number Theorem

- $\pi(x) = \# \{ n \leq x \mid n \text{ is prime} \}$

- $Li(x) = \int_2^x \frac{1}{\ln(t)} \, dt$

- $\pi(x) \sim Li(x)$ as $x \to \infty$

- This is not proved by finding an explicit antiderivative of $\frac{1}{\ln(t)}$.

- If $u = \ln(t)$, then $\int \frac{1}{\ln(t)} \, dt = \int \frac{e^u}{u} \, du$. 
Elementary formulas

- The indefinite integrals $\int e^{-u^2} \, du$ and $\int \frac{e^u}{u} \, du$ do not have elementary formulas.

- How does one prove such claims?

- First have to give a precise definition of “elementary formula”.

- After all $\int e^{-u^2} \, du = \int_a^u e^{-x^2} \, dx + C$ for any constants $a$ and $C$ by FTC.
History

- Newton was perfectly happy to solve an integral by a power series.

- Leibniz preferred integration in "finite terms" and allowed transcendental functions like logarithms.
An elementary function (roughly) should be a function of one variable built out of polynomials, exponentials, logarithms, trigonometric functions, and inverse trigonometric functions, by using the operations of addition, multiplication, division, root extraction, and composition.

Example: \[
\frac{\sin^{-1}(x^3 - 1)}{\sqrt{\ln x + \cos(x / x^2 + 1)}}
\]
A simplification

- We will use $\mathbb{C}$-valued functions of the **real** variable $x$, i.e., our constants will be complex numbers.

- All trigonometric functions and inverse-trigonometric functions can be written in terms of complex exponentials and logarithms.

  $$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

  $$\tan^{-1}(x) = \frac{1}{2i} \left( \ln\left(\frac{x - i}{x + i}\right) - i\pi \right)$$

  $$\sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right), \quad \cos^{-1}(x) = \tan^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right)$$
Meromorphic functions

- A **meromorphic function** is a function defined on an open interval $I$ of the real numbers whose values are complex numbers or $\infty$ with the property that sufficiently close to any $x_0$ in $I$ the function is given by a convergent Laurent series in $x - x_0$.

- Rational functions are meromorphic on $\mathbb{R}$.

- Given a meromorphic function $f$, both $e^f$ and $\ln f$ are meromorphic (one may have to restrict the domain of $f$).
Fields of meromorphic functions

- Let $\mathbb{C}(x)$ denote the field of rational functions. Notice that this field is closed under differentiation.

- Any elementary function (under our rough definition) should be in some “extension” of $\mathbb{C}(x)$. 
Fields of meromorphic functions

- If $f_1, \ldots, f_n$ are meromorphic functions, let $\mathbb{C}(f_1, \ldots, f_n)$ denote the set of all meromorphic functions $h$ of the form
  \[ h = \frac{p(f_1, \ldots, f_n)}{q(f_1, \ldots, f_n)} \]
  for some $n$-variable polynomials $p, q \neq 0$ and $q(f_1, \ldots, f_n)$ is not identically zero.

- This definition captures the operations of addition, multiplication, and division.

- It is not hard to show that the set $\mathbb{C}(f_1, \ldots, f_n)$ is a field and that this field is closed under differentiation.

- Example: $K = \mathbb{C}(x, \sin x, \cos x) = \mathbb{C}(x, e^{ix})$. 
A field $K$ is an **elementary field** if $K = \mathbb{C}(x, f_1, \ldots, f_n)$ and each $f_j$ is

- an exponential or logarithm of an element of $K_{j-1} = \mathbb{C}(x, f_1, \ldots, f_{j-1})$,

- or $f_j$ is **algebraic** over $K_{j-1}$, that is $f_j$ is a solution to an equation $g_l t^l + \cdots + g_1 t + g_0 = 0$ where $g_0, g_1, \ldots, g_l \in K_{j-1}$

An elementary field is built from the the field of rational functions in finitely many steps by adjoining an exponential, a logarithm, or a solution to a polynomial.

Composition is captured by adjoining exponentials or logarithms. Root extraction is captured by the adjunction of algebraic solutions.

Elementary fields are closed under differentiation.
A meromorphic function $f$ is an **elementary function** if it lies in some elementary field.

**Example:** $f(x) = \sqrt[3]{\ln x + \cos \left( \frac{x}{x^2 + i} \right)}$ is an elementary function.

\[
\mathbb{C}(x) \subset \mathbb{C}(x, \ln x) \subset \mathbb{C}(x, \ln x, e^{i \left( \frac{x}{x^2 + i} \right)}) \subset \mathbb{C}(x, \ln x, e^{i \left( \frac{x}{x^2 + i} \right)}, f)
\]
A meromorphic function $f$ can be integrated in elementary terms if $f = g'$ for some elementary function $g$.

Recall an elementary field is closed under differentiation so if $f$ can be integrated in elementary terms, then necessarily $f$ is also elementary.
Differential Galois theory

- We can rephrase our problem: Given an elementary function $f$, when does the differential equation $\frac{dy}{dx} - f = 0$ have an elementary solution?

- The answer is in the affirmative precisely when we can find a tower of fields with special properties.

- Consider the analogy with ordinary Galois theory.
Liouville’s Theorem

- **Theorem (Liouville, 1835):** Let $f$ be an elementary function and let $K$ be any elementary field containing $f$. If $f$ can be integrated in elementary terms then there exist nonzero $c_1, \ldots, c_n \in \mathbb{C}$, nonzero $g_1, \ldots, g_n \in K$, and an element $h \in K$ such that

\[
f = \sum c_j \frac{g_j'}{g_j} + h'.
\]

- If $f = \sum c_j \frac{g_j'}{g_j} + h'$, then $g = \sum c_j \ln(g_j) + h$ is an elementary antiderivative of $f$.

- The theorem is proved by induction on the length of a tower of fields constructing $K(g)$ where $g$ is an antiderivative of $f$. 
**An important corollary**

- **Corollary:** Let $f$ and $g$ be in $\mathbb{C}(x)$ with $f \neq 0$ and $g$ nonconstant. If $f(x)e^{g(x)}$ can be integrated in elementary terms then there is a function $R(x)$ in $\mathbb{C}(x)$ such that $R'(x) + g'(x)R(x) = f(x)$.

- If $R(x) \in \mathbb{C}(x)$ satisfies $R'(x) + g'(x)R(x) = f(x)$, then $R(x)e^{g(x)}$ is an antiderivative of $f(x)e^{g(x)}$.

- We can apply this corollary to show that $e^{-x^2}$ and $e^x/x$ have no elementary antiderivatives.
Proof for $e^{-x^2}$

- Taking $f = 1$ and $g = -x^2$ in the Corollary, we must show the differential equation

$$R'(x) - 2xR(x) = 1 \quad (*)$$

has no solution for $R(x) \in \mathbb{C}(x)$.

- ODE’s shows the general solution of $(*)$ is

$$R(x) = e^{x^2} \left( \int e^{-x^2} \, dx + c \right)$$

for any $c \in \mathbb{C}$ ... but this doesn’t help!
Proof for $e^{-x^2}$

- Suppose that $R(x) \in \mathbb{C}(x)$ is a solution to $(*).$

- $R$ cannot be a constant or a polynomial in $x$ (by degree considerations).

- Write $R(x) = \frac{p(x)}{q(x)}$ for some nonzero relatively prime polynomials $p(x), q(x)$ with $q(x)$ nonconstant.

- Let $z_0 \in \mathbb{C}$ be a root of $q(x)$ of multiplicity $\mu \geq 1$. Then $p(z_0) \neq 0$ and $p(x)/q(x) = h(x)/(x - z_0)^\mu$ with $h(x) \in \mathbb{C}(x)$ having numerator and denominator that are non-vanishing at $z_0.$
Proof for $e^{-x^2}$

- The quotient rule yields

$$\left(\frac{p(x)}{q(x)}\right)' = \frac{-h(x)}{\mu(x - z_0)^{\mu+1}} + \frac{h'(x)}{(x - z_0)^\mu}$$

- As $z \to z_0$ in $\mathbb{C}$ the absolute value of $(p(x))/q(x))'|_{x=z}$ blows up like $A/|z - z_0|^{\mu+1}$ with $A = |h(z_0)/\mu| \neq 0$.

- $| - 2z \cdot (p(z)/q(z))|$ has growth bounded by a constant multiple of $1/|z - z_0|^{\mu}$ as $z \to z_0$.

- Therefore

$$|((p(x)/q(x))' - 2x \cdot (p(x)/q(x)))|_{x=z} | \sim \frac{A}{|z - z_0|^{\mu+1}}$$

as $z \to z_0$.

- This contradicts the identity $R'(x) - 2xR(x) = 1$. 
References


