Fat Point Modules over a Generalized Laurent Polynomial Ring

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Introduction

Fix K, algebraically closed field, char$K = 0$.

Definition (AS). *An $\mathbb{N}$-graded, connected, K-algebra A is AS-regular of dimension d if:*

1. *A has global dimension d.*

2. *A has finite GK-dimension.*

3. *A satisfies the Gorenstein condition:*

$$\mathit{Ext}_n({}_AT, A) = \delta_{n,d} T_A(e)$$

$T_A(e)$ denotes the trivial A-module shifted by $e \in \mathbb{Z}$.

• AS-regular algebras, dimension 2; easily classified

• AS-regular algebras, dimension 3; generated in degree 1 classified by Artin, Schelter, Tate, Van den Bergh, not generated in degree 1, classified by Stephenson

• AS-regular algebras, dimension 4; not classified, many examples studied:

1. Sklyanin algebras

2. regular Clifford algebras

3. normal extensions of 3-dimensional AS-regular algebras

4. graded regular Ore extensions

5. others

• Lu, Palmieri, Wu, Zhang recently classified some Noetherian, 2-generated, AS-regular algebras of dimension 4 using theory of A_∞-algebras

Definition. *A quantum* $\mathbb{P}^3$ *is an AS-regular algebra* A *of dimension 4 whose Hilbert series is* $\frac{1}{(1-t)^4}$*. In particular,* A *can be presented as an algebra on 4 linear generators and 6 quadratic relations.*

Geometry

Associated to any AS-regular algebra A of dimension n

- point modules; graded, cyclic, generated in degree 0, Hilbert series $1/(1-t)$

- if $n \leq 4$, there is a projective scheme Γ, which parametrizes point modules, furthermore Γ is the graph of an automorphism τ

- (Γ, τ); *point scheme* of A

- for a quantum $\mathbb{P}^3$, $\Gamma = \mathcal{V}(\text{rels.})$, subscheme of $\mathbb{P}^3 \times \mathbb{P}^3$

Theorem (ATV). *Let A be an AS-regular algebra of dimension 3. A is a finite module over its center if and only if τ has finite order.*

Does a similar theorem hold in dimension 4?

Theorem (StV, CGSh). *There exist quantum $\mathbb{P}^3$'s where τ has finite order and yet the algebra is not finite over its center.*

Perhaps the action of τ on another set of "geometric" modules detects "not finite over the center"?

More Geometry: Fat Point Modules

• Proj$A$; category of finitely generated A-modules modulo finite length graded A-modules

• Shift functor s : Proj$A \rightarrow$ Proj$A$, $s([M]) = [M(1)]$, corresponds to τ when restricted to point modules

• fat point module of multiplicity $m > 1$; graded, generated in degree 0, Hilbert series $m/(1-t)$, critical with respect to GK-dimension

• fat point; $[F] \in$ Proj$A$ for fat point module F, simple object in Proj$A$

Theorem (GSh). *The generic member of StV examples has a one-parameter family of fat points of multiplicity 2 upon which s has infinite order.*

Choose $r, \alpha \in K^\times$.

- $A(r)$, down-up algebra; $K\langle x, y\rangle$ factored by $\langle x^2y-(r+r^{-1})xyx+yx^2, y^2x-(r+r^{-1})yxy+xy^2\rangle$

- $C(r,\alpha) = A(r)[d,u;\sigma,h_r]$ where $\sigma = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$ and $h_r = (xy - ryx)$

- $C(r,\alpha)$ free $A(r)$-module on basis $\{d^i, i \geq 0\} \cup \{u^j, j > 0\}$

Theorem (CGSh). *$C(r,\alpha)$ is a Noetherian domain and a quantum $\mathbb{P}^3$.*

Theorem (CGSh). *Suppose that r is NOT a root of unity and that α IS a primitive k-th root of unity. Let $C = C(r, \alpha)$ then:*

1. *The center of C is*
$$Z(C) = \begin{cases} K[(h_r h_{r-1})^{k/2}, d^{2k}, u^{2k}] & \text{if } k \text{ is even,} \\ K[(h_r h_{r-1})^{k}, d^{2k}, u^{2k}] & \text{if } k \text{ is odd.} \end{cases}$$

2. *C is not finite over its center.*

3. *The point scheme of C is finite (4 points, not reduced) and the associated automorphism has finite order.*

Questions:

1. What are the fat point modules of $C(r, \alpha)$?

2. How does the shift functor s act on the fat point modules?

• for rest of talk, r not a root of unity, α a primitive k-th root of unity, $C = C(r, \alpha)$

• for a graded C-module M, *center acts trivially* on M if $MZ(C)_{>1} = 0$, otherwise *center acts nontrivially*

Theorem. *Let F be a fat point module for C. Then:*

1. *The multiplicity of F is 2.*

2. *The family of fat points for which the center acts trivially is parametrized by $\zeta \in K^\times$ and $s[F(\zeta)] = [F(r\zeta)]$.*

3. *The family of fat points for which the center acts non-trivially is parametrized by $\lambda \in K^\times$ and $s[F(\lambda)] = [F(\alpha^{-1/2}\lambda)]$.*

- proofs use noncommutative algebraic geometry of some cubic AS-regular algebras of global dimension 3

- fix a fat point module, F

- for a normal element $c \in C$, $[Fc]$ is either zero or $[F]$

- d^2, u^2, normal regular elements in $C(r, \alpha)$

Proofs

Case 1: Suppose $Z(C)$ acts trivially on F so that $[Fd^2] = [Fu^2] = 0$.

- $\tilde{F}$, F restricted to $A(r)$, show $\tilde{F}$ has a critical composition series and trivial socle

- $A(r)$ not finite over its center so all critical modules are (shifted) point modules

- $\tilde{F}$ contains a point module for $A(r)$

- prove multiplicity of F is 2, find explicit action of C on F, compute action of s, action has infinite order

Case 2: Suppose $Z(C)$ acts nontrivially on F, then $[Fd^2] = [F]$ or $[Fu^2] = [F]$, i.e. d^2 or u^2 act injectively on F

• localize at d^2 or u^2, take degree 0, obtain Λ_d or Λ_u

• fat point modules where d (resp. u) acts injectively correspond to finite dimensional simple Λ_d (resp. Λ_u) modules

• define $B = K\langle x, y\rangle / \langle \alpha x^3 - \beta(y^2x + xy^2), \alpha^{-1}y^3 - \beta(x^2y + yx^2)\rangle$ where $\beta = r/(r^2+1)$, B is AS-regular of dimension 3, $\Lambda_d \cong \Lambda_u \cong B$

remark: B is not isomorphic to $A(r)$

• V finite-dimensional simple B-module, B is not finite over its center, so can show V is a quotient of a point module of B

• $\dim V = 1$ or 2, multiplicity of F is 2, explicit action of C on F, action has finite order

Conjecture. Let A be a quantum $\mathbb{P}^3$. Suppose s has finite order on the scheme of point modules and finite order on all fat points. Then A is a finitely generated module over its center.

Questions:

1. What happens when r is a root of unity or α isn't a root of unity?

2. If A is finitely generated over its center, does s have finite order on the point scheme and finite order on the fat points?

3. Are the fat point modules (of fixed multiplicity) parametrized by a (projective) scheme?

More examples needed!