Bezout's Theorem: A Tale of Two Curves

Math Colloquium
Seattle University
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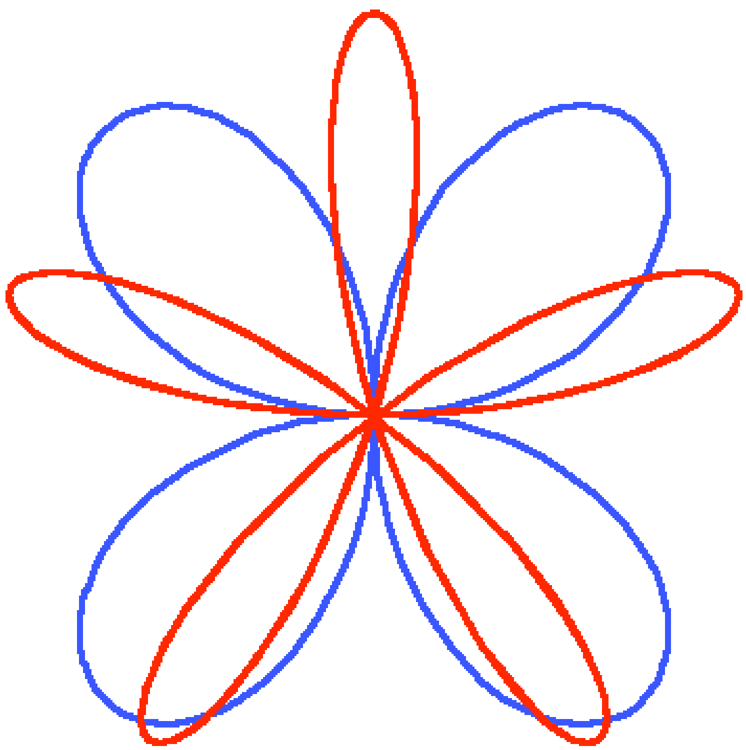

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Outline

I. Introduction and Examples.

II. Main Problems.

III. Algebraic Geometry.

IV. Bezout's Theorem.

I. Introduction and Examples

• Let $f(x, y)$ be a polynomial over $\mathbb{C}$.

Definition. *The degree of f is the maximum of the degrees of the monomials appearing in f.*

Ex. $\deg(7 + 3xy^6 - 5x^7 + 2x^6y^7) = 13$.

• $f(x, y)$ yields a function on the complex plane, $\mathbb{C}^2$.

• The zeroes of $f(x, y)$ give an algebraic curve C in $\mathbb{C}^2$,

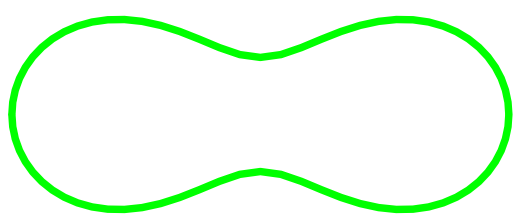
$$C = \left\{(a, b) \in \mathbb{C}^2 \mid f(a, b) = 0\right\}.$$

• Given an algebraic curve $C : f(x, y) = 0$, define $\deg(C) = \deg(f)$.

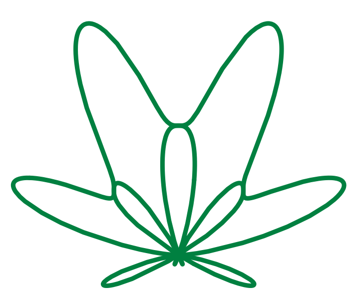
Here are some interesting examples.

Ex. folium of Descartes: $x^3 + y^3 = 6xy$

Ex. oval of Cassini: $(x^2+y^2+9)^2-36x^2 = 100$

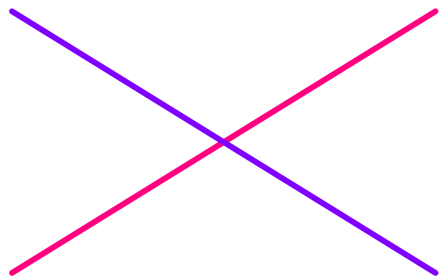


Ex. frog curve: $r = \sin\theta + \sin^3 4\theta$, deg $= 24$

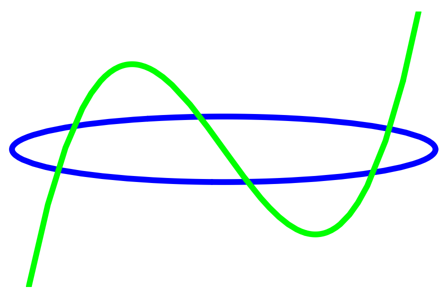


Problem: Given two algebraic curves C and D, count the number of points in $C \cap D$.

Ex. two lines: $1 = 1 \cdot 1$

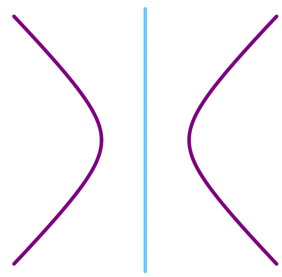


Ex. ellipse and cubic: $6 = 2 \cdot 3$

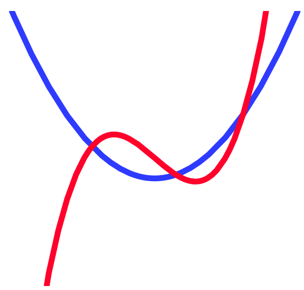


Elegant Guess: Let $\deg(C) = e$, $\deg(D) = f$. If C and D don't have any common components then $|C \cap D| = ef$.

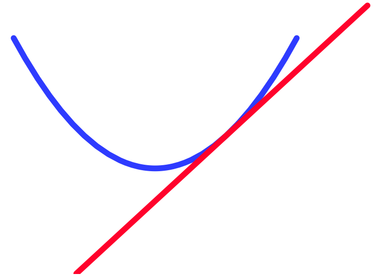
Ex. line and hyperbola: $0 \neq 1 \cdot 2$



Ex. parabola and cubic: $3 \neq 2 \cdot 3$



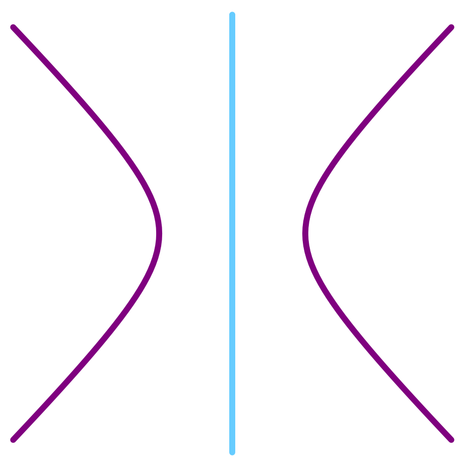
Ex. parabola and tangent line: $1 \neq 2 \cdot 1$



II. Main Problems

A. Solutions in $\mathbb{C}$

Ex. $x^2 - y^2 = 1$, $x = 0$.

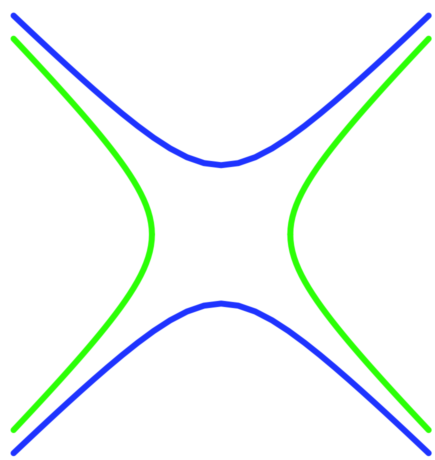


• Intersecting implies $y^2 = -1$ so we have 2 points: $(0, \pm i)$.

• Therefore we should count points in $\mathbb{C}^2$ and remember that we can only draw pictures in $\mathbb{R}^2$.

B. Points at Infinity and $\mathbb{P}^2$

Ex. Two hyperbolas: $x^2 - y^2 = 1$, $y^2 - x^2 = 1$.



• It is easy to check that these two hyperbolas share no complex points. Where might we find the missing 4 points?

Idea: We can try to add points at ∞ corresponding to directions.

The Complex Projective Plane

Definition. *The complex projective plane* $\mathbb{P}^2$ *is defined by*

$$\mathbb{P}^2 = \left\{ [a,b,c] \mid (a,b,c) \in \mathbb{C}^3 \setminus \{(0,0,0)\} \right\}$$

where $[a,b,c] = [a',b',c']$ *if and only if there exists* $0 \neq \lambda \in \mathbb{C}$ *such that* $(a',b',c') = \lambda(a,b,c)$.

• If $c \neq 0$, then $[a,b,c] = [a/c, b/c, 1]$.

• There is a one-to-one correspondence between points in $\mathbb{P}^2$ of the form $[a',b',1]$ and points in $\mathbb{C}^2$.

• We say the point $[a,b,0]$ lies on the line at infinity $z = 0$.

• $[a,b,0]$ is the point at ∞ corresponding to the direction of the line through the origin of slope b/a.

Polynomial Functions on $\mathbb{P}^2$

- Let $R = \mathbb{C}[x, y, z]$.

Ex. Let $f(x, y, z) = x + y^2 + z^3$ and $p = [1, 1, 2] = [2, 2, 4]$. Then $f([1, 1, 2]) = 10$ and $f([2, 2, 4]) = 70$???

Problem: $f(x, y, z) \in R$ does not give a well defined function on $\mathbb{P}^2$.

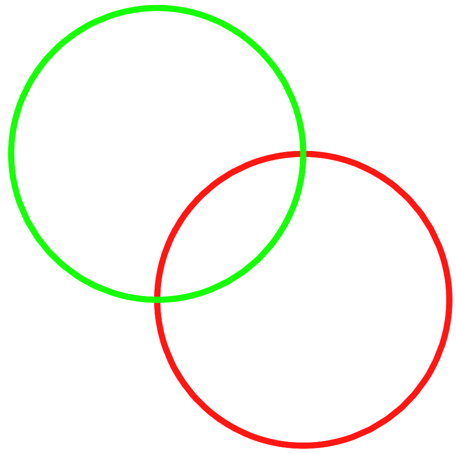
Definition. *A homogeneous polynomial (in three variables) of degree n is a polynomial $f(x, y, z)$ such that $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$ for all $\lambda \neq 0$.*

Ex. $x^3 + xy^2 - yz^2 + 2z^3$ is homogeneous of degree 3.

- The **zeroes** of homogeneous polynomials are well-defined in $\mathbb{P}^2$.

- A curve in $\mathbb{P}^2$ or a projective curve is the zero set of a **homogeneous** polynomial.

Ex. Two circles, $C : (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$, and $D : x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$.



• Visually there are two points of intersection in $\mathbb{C}^2$.

Homogenize:

$C : (x - \frac{1}{2}z)^2 + y^2 = \frac{1}{4}z^2$.

$D : x^2 + (y - \frac{1}{2}z)^2 = \frac{1}{4}z^2$.

• $z = 0 \Rightarrow x^2 + y^2 = 0$ giving points $[1, \pm i, 0]$

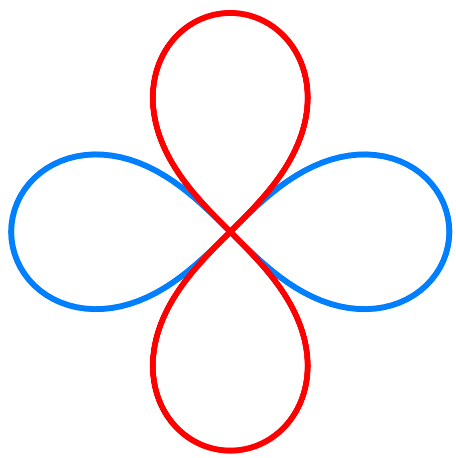
• $4 = 2 \cdot 2$

C. Counting Multiplicities

Ex. lemniscates of Bernoulli

$C : (x^2 + y^2)^2 - (x^2 - y^2) = 0,$

$D : (x^2 + y^2)^2 - (y^2 - x^2) = 0$



- The only complex point of intersection is $(0,0)$. There are only two points at ∞, $[1, \pm i, 0]$.

- We would like to count $4 \cdot 4 = 16$ points so have to account for "multiplicities".

III. Algebraic Geometry

- Let $R = \mathbb{C}[x,y]$.
- For $X \subset \mathbb{C}^2$, let

$$I(X) = \Big\{ f(x,y) \in R \mid f(a,b) = 0, \forall (a,b) \in X \Big\}.$$

- For $I \subset R$, let

$$V(I) = \Big\{ (a,b) \in \mathbb{C}^2 \mid f(a,b) = 0, \forall f(x,y) \in I \Big\}.$$

Remarks:

1. The sets of the form $V(I)$ form the closed sets of a topology on $\mathbb{C}^2$.
2. Hilbert's Nullstellensatz:

$$\Big\{ \text{algebraic sets, } V(I) \Big\} \rightleftarrows \Big\{ \text{radical ideals, } \sqrt{I} \Big\}.$$

- Given $X \subset \mathbb{C}^2$, what are the polynomial functions on X?

Definition. *Let $X \subset \mathbb{C}^2$. The coordinate ring of X is $A[X] = R/I(X)$.*

• If $f - g \in I(X)$ then $f = g$ as functions on X.

• The coordinate ring $A[X]$, and algebraic geometry enable us to define multiplicities.

Definition. *Let C and D be two projective curves, $p \in C \cap D$.*

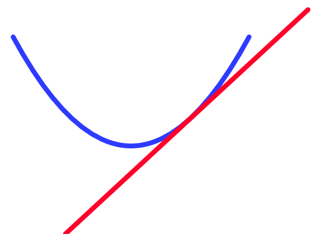
1. *The intersection multiplicity of p in $C \cap D$ is $I(p, C, D) = \dim_{\mathbb{C}}(R/I(C) + I(D))_p$.*

2. *The multiplicity of p in C, denoted $m(p, C)$, is the multiplicity of the Hilbert-Samuel polynomial of $A[C]_p$.*

• Equivalently, $m(p, C)$ is the "order of vanishing" of p in C.

Ex. $m((0,0), y^2 - x^3)) = 2$.

Ex. parabola and tangent line, $y = x^2$, $y = 2x - 1$



- $\mathbb{C}[x, y]/(y - x^2, y - 2x + 1) \cong \{a + bx \mid a, b \in \mathbb{C}\}$

as vector spaces.

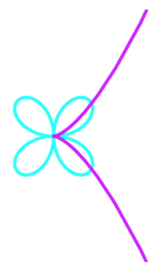
- $\dim\{a + bx \mid a, b \in \mathbb{C}\} = 2 = 2 \cdot 1$.

Facts:

1. (Deformation) $I(p, C, D) = I(p, C, D + AC)$ for any curve A.

2. (Transversality) $I(p, C, D) \geq m(p, C)m(p, D)$ with equality if and only if C and D meet transversely at p.

Ex. cusp and 4 leaf clover, $C : y^2 = x^3$, $D : (x^2 + y^2)^3 = 4x^2y^2$



Intersect:

Substitute $y^2 = x^3$ into $(x^2 + y^2)^3 = 4x^2y^2$ to get $x^6(x + 1)^3 = 4x^5$.

Hence $x^5(x(x + 1)^3 - 4) = 0$.

There are 4 distinct roots of $x(x + 1)^3 - 4$.

For each such root we get two points of intersection from $y^2 = x^3$.

So there are 8 complex points other than the origin.

- Let $p = (0, 0)$.

- What is the multiplicity of p?

Deform:

Consider $D - (x^6 + 3x^5 + 3x^4 - 4x^2 + 3x^2y^2 + x^3y^2 + y^4)C = x^5(x(x+1)^3 - 4) = D'$.

Now the clover D has been deformed to 5 lines of the form $x = \lambda$.

The line $x = 0$ meets C transversely at p.

Hence

$I(p, C, D) = I(p, C, D') = m(p, C) \cdot m(p, D') = 2 \cdot 5 = 10.$

- $10 + 8 = 18 = 3 \cdot 6 = \deg(C) \cdot \deg(D)$.

Theorem (Bezout). *Let C and D be projective curves having no common component. Let C and D have degrees e and f respectively. Then, counting multiplicity,*

$$|C \cap D| = ef.$$

History:

1. Maclaurin, Euler, Cramer (1700's) assert the theorem , no valid proof.

2. Etienne Bezout (1730-1783), flawed proof, didn't account for multiplicities correctly.

3. Georges-Henri Halphen (1873), first correct proof.