The Fundamental Group, Braids and Circles

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Outline

1) Topology

2) Path Homotopy and The Fundamental Group

3) Covering Spaces and The Fundamental Group of The Circle

4) Brouwer Fixed Point Theorem; Perron-Frobenius Theorem

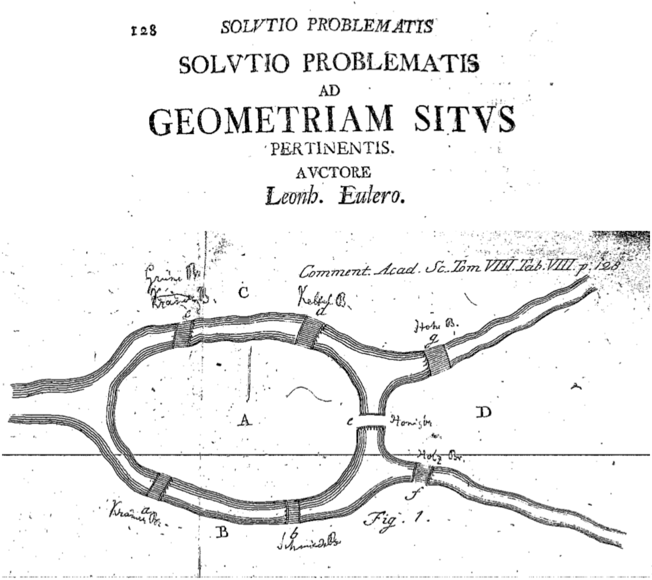
5) The Braid Group

In the beginning...

Leonhard Euler
(15 April 1707 - 18 September 1783)

The city of Königsberg, Prussia (now Kaliningrad, Russia) lies on both sides of the Pregola River and includes two islands that were connected to each other and the mainland by seven bridges. The problem was to find a walk through the city that crossed each bridge once and only once. A solution didn't require the starting and ending spots to be the same.

Geometriam Situs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The bridges of Königsberg

Euler's solution

Euler (1735) argued such a walk was impossible as follows. If such a walk was possible, then

1) each vertex besides the starting and ending points must have even degree.

2) In the Königsberg graph all vertices have odd degree: (3, 3, 3, 5).

3) Since at most two vertices can serve as the endpoints of a walk, it is impossible to make such a walk.

In fact, as Euler stated, such a walk in a connected graph is possible if and only if there are zero or two vertices of odd degree.

We now call such a walk an *Eulerian path*.

A new subject

Euler had discovered a new branch of mathematics, called topology.

In topology, distances, area, angles and other notions we are familiar with from geometry take a back seat.

Euler's solution of the Königsberg problem shows that the pertinent components are: the relative positions of the land masses, and the connections that the bridges make.

Originally the subject was called "Analysis Situs" (Leibniz), or the analysis of position. The term "topology" was probably first used at the beginning of the 20th century.

Topology Definition

Topology is the study of properties of spaces that are preserved under continuous maps.

What are spaces?

What are continuous maps?

What are properties that are preserved?

Spaces

Definition

Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X called *open sets* such that

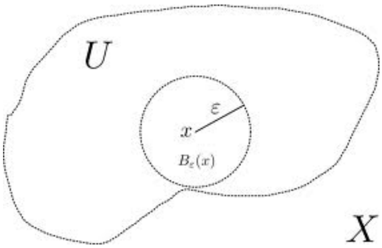
1) $\emptyset$ and X are open sets,
2) the union of any collection of open sets is open,
3) the intersection of two open sets is open.

This definition was probably first introduced by Bourbaki, but there were many precursor definitions; Hausdorff, Sierpínski for example.

A **topological space** is a set X endowed with a topology.

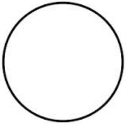
Topology on $\mathbb{R}^n$

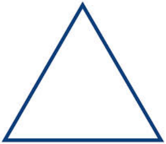
A topology on $X = \mathbb{R}^n$ is familiar from calculus: a set U in $\mathbb{R}^n$ is **open** if every point in U can be surrounded with a small enough open ball which is entirely contained in U.



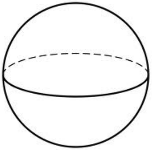
Topological Spaces in $\mathbb{R}^2$, and $\mathbb{R}^3$

In $\mathbb{R}^2$:





In $\mathbb{R}^3$:



Continuous Maps between Spaces

A function $f : X \to Y$ between topological spaces X and Y is **continuous** if for every open set U of Y, the preimage of U, $f^{-1}(U)$ is open in X.

For subspaces of $\mathbb{R}^n$ this definition is equivalent to everyone's favorite:

$f : X \to Y$ is continuous if for every x_0 in X and all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Homeomorphism

When are two topological spaces considered equivalent?

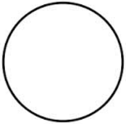
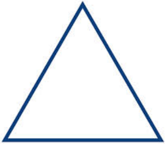
Remember we can stretch and bend spaces as long as we don't tear, or punch holes in them, for example. (Actually, you can tear and cut spaces, as long as you glue them back together in a "continuous" fashion.)

A **homeomorphism** from X to Y is a continuous bijective map $f : X \to Y$ such that $f^{-1} : Y \to X$ is also continuous.

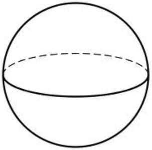
Write $X \cong Y$ for this equivalence relation.

Homeomorphic and Non-homeomorphic spaces

In $\mathbb{R}^2$:

 $\cong$  $\ncong$ 

In $\mathbb{R}^3$:

 $\ncong$  $\ncong$ 

Jokes

Or....

A topologist, a person who cannot tell his behind from a hole in the ground, but can tell his behind from two holes in the ground.

$\not\cong$

Topological Invariants

In general, it can be a hard problem to decide if two topological spaces are homeomorphic.

One tries to find a property of topological spaces that is preserved under homeomorphism. Such a property is called a **topological invariant**.

Some properties you might know from analysis: **compactness**, **connectedness**, **path-connectedness**.

None of these properties serve to distinguish the circle from the solid square, nor the sphere from the torus.

We need a finer invariant.

Henri Poincaré and The Fundamental Group

Henri Poincaré
(29 April, 1854 - 17 July, 1912)

Poincaré, in his investigations of singularities in differential equations on spaces, constructed a **group** associated to any topological space. The elements of this group will be “loops” in the space, and the operation will be “go around the first loop, then go around the second loop.”

Loops

Let X be a space, and fix a point x_0, called the basepoint.

(For experts, we'll assume that X is path-connected so that the group we're about to build doesn't depend on this choice.)

A **loop** in X based at x_0 is a continuous map

$$f : [0,1] \to X$$

such that $f(0) = f(1) = x_0$.

For two loops f, g we define their product $f * g : [0,1] \to X$ by

$$f * g(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq 1/2 \\ g(2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Is the set of all loops at x_0 with this operation a group?

Group?

The identity element should be the constant loop e_{x_0};
the inverse of a loop should be $\overline{f} =$ "run f backwards" .

But then there's associativity:

$$f * (g * h) \neq (f * g) * h.$$

And then thinking about what we said above,

$$f * e_{x_0} \neq f, \quad f * \overline{f} \neq e_{x_0}.$$

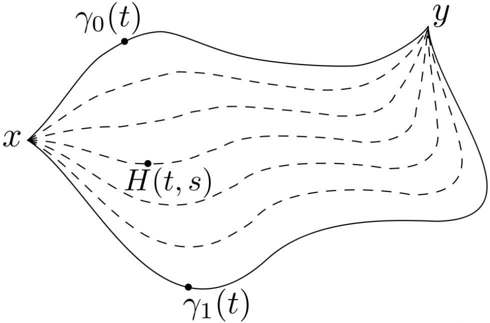
But there should be a way to reparametrize....

Path Homotopy

We'll slightly generalize and consider **paths** from x to y.
Let γ_0 and γ_1 be paths from x to y.
A **path homotopy** from γ_0 to γ_1 is a continuous map $H : I \times I \to X$ such that

$$H(t,0) = \gamma_0(t), H(t,1) = \gamma_1(t); H(0,s) = x, H(1,s) = y.$$

Write $\gamma_0 \sim \gamma_1$.



The Fundamental Group

For a loop f at x_0, let $[f]$ be its homotopy class (equivalence class). Let $\pi_1(X)$ be the set of all $[f]$. Then:

1) $[f] * [g] = [f * g]$ is well-defined,

2) $[f * (g * h)] = [(f * g) * h]$,

3) $[f * e_{x_0}] = [e_{x_0} * f] = [f]$,

4) $[f * \overline{f}] = [\overline{f} * f] = [e_{x_0}]$.

Therefore $\pi_1(X)$ is a group under the operation $*$. This is the **fundamental group** of X.

π_1 is a functor

Let $g : X \to Y$ be a continuous map. Define
$g_* : \pi_1(X) \to \pi_1(Y)$ by $g_*([f]) = [g \circ f]$.

Then g_* is a homomorphism of groups. Moreover,

$$(g \circ h)_* = g_* \circ h_* \text{ and } (\mathrm{id})_* = \mathrm{id}.$$

Therefore π_1 is a **functor** from the category of topological spaces to the category of groups.

A key point: if X and Y are homeomorphic spaces, then $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic groups.

"Algebraic Topology can be roughly defined as the study of techniques for forming algebraic images of topological spaces." - Allen Hatcher

He goes on to describe such functors as 'lanterns'.

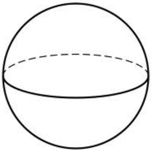
Distinguishing some surfaces

$\pi_1(\text{2-sphere}) = 0$

$\pi_1(\text{torus}) = \mathbb{Z} \times \mathbb{Z}$

$\pi_1(\text{2-holed torus}) = (\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z})/N$

Therefore



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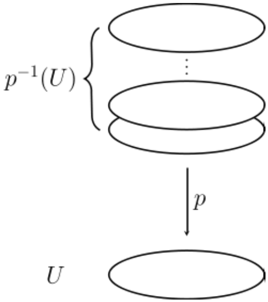
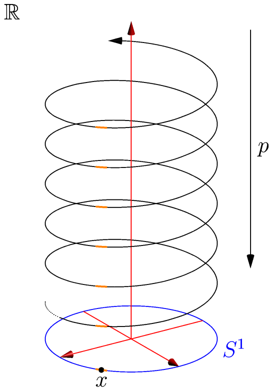
Covering spaces

A **covering space** E for a space B consists of a continuous, surjective map $p : E \to B$ such that for each point $x \in B$ there exists an open set Ucontaining x with $p^{-1}(U)$ homeomorphic to a disjoint union of copies of U.





Path lifting

Let $p : E \to B$ be a covering map.

Let $f : [0,1] \to B$ be a path in B with $f(0) = x_0$; choose a point $e_0 \in p^{-1}(x_0)$.

Then there is a unique way to "lift" the path f to a path $\tilde{f}$ in E with $\tilde{f}(0) = e_0$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

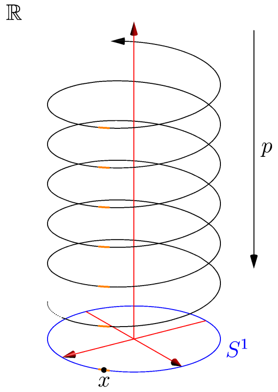
In this way we get a map

$$\phi : \pi_1(B, x_0) \to p^{-1}(x_0), \quad \phi[f] = \tilde{f}(1).$$

If E is simply connected (E is path connected and $\pi_1(E) = 0$), then ϕ is bijective.

Fundamental group of S^1

Define $p : \mathbb{R} \to S^1$ by $p(x) = (\cos(2\pi x), \sin(2\pi x))$.
Then p is a covering map.

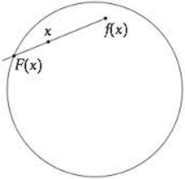


If we take our basepoint to be $(1,0)$ in S^1, then $p^{-1}(1,0) = \mathbb{Z}$.
Then you can show the map $\phi : \pi_1(S^1,(1,0)) \to \mathbb{Z}$ is an isomorphism!

Brouwer fixed point theorem for D^2

Theorem: Any continuous function $f : D^2 \to D^2$ has a fixed point.

Proof: Suppose not, and let $f : D^2 \to D^2$ be a continuous map with no fixed point. Then the map F is a continuous map $D^2 \to S^1$.



Also note that F is the identity when restricted to S^1. Let $i : S^1 \to D^2$ be the inclusion map. Then the composite $F \circ i$ is the identity map on S^1. We have then a sequence of homomorphisms

$$\pi_1(S^1) \xrightarrow{i_*} \pi_1(D^2) \xrightarrow{F_*} \pi_1(S^1)$$

$$\mathbb{Z} \xrightarrow{\quad i_* \quad} 0 \xrightarrow{\quad F_* \quad} \mathbb{Z}$$

which is a contradiction.

Perron-Frobenius Theorem in $\mathbb{R}^3$

Theorem: Let A be a 3×3 real matrix with positive entries. Then A has a positive real eigenvalue.

Proof: Let X be the intersection of the unit sphere with the first octant in $\mathbb{R}^3$. Then it is easy to see that X is homeomorphic to the disk D^2.

Any space which is homeomorphic to the disk has the fixed point property, so X has the fixed point property.

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point in X. Then $A\mathbf{x}$ is a vector whose components are all positive. Therefore the map $X \to X$ given by $\mathbf{x} \mapsto A\mathbf{x}/||A\mathbf{x}||$ has a fixed point.

Let $\mathbf{y}$ be a fixed point. Then $\mathbf{y} = A\mathbf{y}/||A\mathbf{y}||$, so

$$A\mathbf{y} = ||A\mathbf{y}||\mathbf{y}.$$

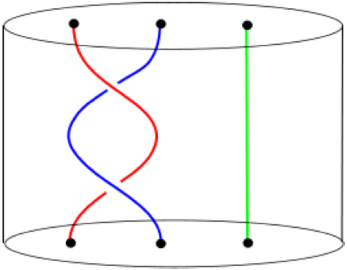
That is, $\mathbf{y}$ is an eigenvector for the positive real eigenvalue $||A\mathbf{y}||$.

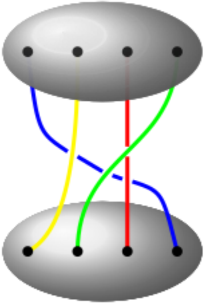
Braid Group of $\mathbb{R}^2$

The **configuration space** of n points in $\mathbb{R}^2$ is the space of all n-tuples $(x_1, x_2, \ldots x_n)$ where each x_i is a point in $\mathbb{R}^2$ and none of the points are equal.

We also identify $(x_1, x_2, \ldots, x_n)$ with $(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for any permutation σ.

Then we can define a **braid** on n strands as an element of the fundamental group of this configuration space.





Thank you for listening!

References

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2) A. Hatcher, *Algebraic Topology*. Cambridge University Press. (Available for free download from Dr. Hatcher's website.)

3) J. Munkres, *Topology (Second Edition)*. Prentice Hall, Inc..