Some non-Koszul algebras from rational homotopy theory

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AMS Fall Western Sectional Meeting
San Francisco State University
October 25, 2014

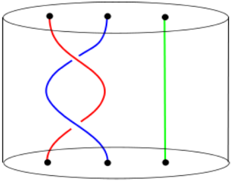
Overview

1. Introduction
2. Cohomology of $P\Sigma_n$
3. Koszulity
4. Ongoing work

Pure braid group

$C_n = \mathrm{Conf}_n(\mathbb{R}^2) = \{(z_1, \ldots z_n) \mid z_i \neq z_j \text{ for } i \neq j\}$

$P_n := \pi_1(C_n, \overline{z_0})$



E. Artin:

• there exists a nice presentation of P_n

• P_n is isomorphic to a subgroup of $\mathrm{Aut}(F_n)$, where F_n is a free group of rank n.

Pure string motion group

R_n: configuration space of n unlinked circles in $\mathbb{R}^3$

$P\Sigma_n$: fundamental group of R_n

$P\Sigma_n$, in this guise, has been studied by Dahm (1962), Goldsmith (1981), Brendle-Hatcher (2010).

McCool (1986): determined a presentation of $P\Sigma_n$ as a subgroup of $\mathrm{Aut}(F_n)$

McCool's presentation of $P\Sigma_n$

Generators

$$\alpha_{ij}(x_k) = \begin{cases} x_j x_i x_j^{-1} & \text{if } k = i \\ x_k & \text{if } k \neq i \end{cases}$$

Relations

$$\begin{aligned} [\alpha_{ij}, \alpha_{ik}\alpha_{jk}] & \quad i, j, k \text{ distinct} \\ [\alpha_{ij}, \alpha_{kj}] & \quad i, j, k \text{ distinct} \\ [\alpha_{ij}, \alpha_{kl}] & \quad i, j, k, l \text{ distinct} \end{aligned}$$

Elements of $P\Sigma_n$ have been called *pure symmetric automorphisms* and *basis-conjugating automorphisms*.

A presentation of $H^*(P\Sigma_n;\mathbb{Q})$

Brownstein-Lee (1993): conjectured a presentation of the cohomology algebra $H^*(P\Sigma_n;\mathbb{Q})$

In 2006,

Theorem (Jensen-McCammond-Meier)

Let $n \geq 2$; let E be the exterior algebra over $\mathbb{Q}$ generated in degree 1 by elements a_{ij}, $1 \leq i \neq j \leq n$. Let $I \subset E$ be the homogeneous ideal generated by $a_{ij}a_{ji}$ for all $i \neq j$ and

$$a_{kj}a_{ji} - a_{kj}a_{ki} + a_{ij}a_{ki}$$

for distinct i, j, k. As graded algebras, $H^*(P\Sigma_n, \mathbb{Q}) \cong E/I$. In particular, the Hilbert series of $H^*(P\Sigma_n, \mathbb{Q})$ is $h(t) = (1+nt)^{n-1}$.

The Koszul question

Around 2008,

Question (Cohen-Pruidze)

Is $H^*(P\Sigma_n; \mathbb{Q})$ a Koszul algebra?

Question was motivated by rational homotopy theory.

Koszul property here is closely linked with formality properties of the Eilenberg-Mac Lane space of the group.

Theorem (Kohno)

$H^*(P_n; \mathbb{Q})$ is Koszul.

Koszul definitions

Given a quadratic algebra, A, there exists a canonical quadratic dual algebra, $A^!$.

Definition

A quadratic $\mathbb{Q}$-algebra A is Koszul if $A^! \cong \mathrm{Ext}^*_A(\mathbb{Q}, \mathbb{Q})$ as graded algebras.

The Yoneda algebra, $\mathrm{Ext}^*(\mathbb{Q}, \mathbb{Q})$, is bigraded.

An equivalent definition of Koszul: $Ext^{i,j}_A(\mathbb{Q}, \mathbb{Q}) = 0$ for all $i \neq j$.

A is Koszul $\iff$ $A^!$ is Koszul.

Results

$A_n := H^*(P\Sigma_n; \mathbb{Q})$

$U(\mathfrak{g}_n) := (A_n)^!$

Theorem (Conner-Goetz)

1. *For $n = 2, 3$, $U(\mathfrak{g}_n)$ is Koszul.*
2. *For $n \geq 4$, $U(\mathfrak{g}_n)$ is not Koszul.*

Theorem (Conner-Goetz)

1. *$U(\mathfrak{g}_n)$ is isomorphic to a smash product $T_n \# U(\mathfrak{g}_{n-1})$, where T_n is a certain subalgebra of $U(\mathfrak{g}_n)$.*
2. *T_n is finitely generated, but not finitely related.*

Problem and Question

There is a natural action of the symmetric group, S_n, on $U(\mathfrak{g}_n)_k$ for any fixed $k \geq 1$.

Problem

For fixed $k \geq 1$, decompose $U(\mathfrak{g}_n)_k$ into irreducible S_n-representations.

Question

For fixed $k \geq 1$, is the sequence of S_n-representations $\{U(\mathfrak{g}_n)_k\}_{n\geq 2}$ representation stable?

Paper is available at: arXiv:1407.4726 [math.RA]

Will appear in: The Bulletin of the London Mathematical Society

Thank you for listening!