K_2 algebras and A_∞ structures

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Outline

K_2

Definition

A connected graded k-algebra A, which is finitely generated in degree 1 is a K_2 *algebra* if the associated bigraded Ext algebra, $E(A)$, is generated as an algebra by $E^1(A)$ and $E^2(A)$ (i.e. generated in cohomology degrees 1 and 2 by Yoneda product).

All Koszul algebras are K_2.

Work of T. Cassidy and B. Shelton shows many other interesting classes of algebras are K_2.

e.g.

- Artin-Schelter regular algebras of global dimension 4 on 3 linear generators
- graded complete intersections

Theorem (Cassidy, Shelton)

Let A be a connected graded k-algebra, finitely generated in degree 1 and let $Q_\bullet \to k$ be a minimal projective resolution of k. Then the following are equivalent:

1. *A is K_2*
2. *For $2 < n < pd_A(k)$, Q_n is finitely generated and the rows of a certain matrix are linearly independent over k.*

The certain matrix is built out of the matrices coming from the maps in $Q_\bullet \to k$.

A_∞

An A_∞-*algebra* over the base field k is a $\mathbb{Z}$-graded vector space

$$E = \oplus_{p \in \mathbb{Z}} E^p$$

together with graded k-linear maps

$$m_n : E^{\otimes n} \to E, \quad n \geq 1$$

of degree $2 - n$ (called *higher multiplications*) which satisfy the Stasheff identities:

$$\mathbf{SI(}n\mathbf{)} : \qquad \sum (-1)^{r+st} m_u (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

where the sum runs over all decompositions $n = r + s + t$ $(r, t \geq 0$ and $s \geq 1)$ and $u = r + 1 + t$.

A basic result on A_∞-algebras is the following.

Theorem (Kadeishvili)

Let C be an A_∞-algebra and let $E = HC$ be the cohomology ring of C. There exists an A_∞ structure on E with $m_1 = 0$ and m_2 induced by multiplication on C. Furthermore this A_∞ structure on E is unique up to quasi-isomorphism.

Merkulov's Construction

- $C = (C, \partial, \cdot)$, a dga over k
- $B =$ coboundaries $\qquad Z =$ cocycles
- $E = HC$, cohomology algebra
- Choose k-vector space splittings:

$$Z = B \oplus H$$

$$C = Z \oplus L = B \oplus H$$

- Identify E with H

- $p : C \to C$, projection to H
- Choose a special homotopy $G : C \to C$ from id_C to p, i.e. $id_C - p = \partial G + G\partial$ and

  $G = 0$ on L and H

  $G = \partial^{-1}$ on B
- Define $\{\lambda_n : C^{\otimes n} \to C\}_{n \geq 2}$ recursively by

  λ_2 is multiplication on C

  $G\lambda_1 = -id_C$

$$\lambda_n = \sum_{s+t=n} (-1)^{s+1} \lambda_2(G\lambda_s \otimes G\lambda_t)$$

Theorem (Merkulov)

Let $m_i = p\lambda_i$ for all $i \geq 2$. Then $(E, m_2, m_3, \ldots)$ is an A_∞-algebra.

Theorem (May, Keller)

Let A be a connected graded k-algebra, finitely generated in degree 1. Let $E = Ext_A(k,k)$ be the associated Ext-algebra. Then A is Koszul if and only if the A_∞ structure on E determined by Kadeishvili's theorem has $m_n = 0$ for all $n \neq 2$.

Questions:

1. What limitations does the K_2 condition place on the A_∞ structure on the Ext-algebra?

2. Do certain types of A_∞ structures on the Ext-algebra guarantee the original algebra is K_2?

Computations

- A, connected graded k-algebra
- Fix a minimal free resolution $Q_\bullet \to k$
- Choose matrix representations for the maps in $Q_\bullet$
- Let C be the morphism complex $\mathrm{Hom}_A(Q, Q)$, a dga
- Merkulov's construction yields a canonical A_∞ structure on HC (cohomology of C)
- $E = \mathrm{Ext}_A(k, k)$ is quasi-isomorphic to HC
- Minimality can be used to show there exists a homotopy $G : C \to C$ such that

$$p\lambda_n = -p\lambda_2(1 \otimes G\lambda_{n-1}) \quad \text{for all } n \geq 2$$

where $p : C \to HC$ is a projection

- Recall that $m_n = p\lambda_n$

Computing a value of m_3

Let $\alpha \in E^p$, $\beta \in E^q$, and $\gamma \in E^r$.

Compute $m_3(\alpha, \beta, \gamma)$.

Since $\deg(m_3) = -1$, $m_3(\alpha, \beta, \gamma) \in E^{p+q+r-1}$.

The class $m_3(\alpha, \beta, \gamma)$ is represented by a map $Q_{p+q+r-1} \to k$.

$$m_3(\alpha,\beta,\gamma) = \pm p\lambda_2(\alpha\otimes G\lambda_2(\beta\otimes\gamma))$$

$$\begin{tikzcd}
Q_{p+q+r} \arrow[r] \arrow[d, "\gamma"] & Q_{p+q+r-1} \arrow[r] \arrow[d, "\gamma"] \arrow[ddl, dashed] & \cdots \arrow[r] \arrow[d, phantom, "\cdots"] & Q_{r+q+2} \arrow[r] \arrow[d, "\gamma"] & Q_{r+q+1} \arrow[r] \arrow[d, "\gamma"] \arrow[ddl, dashed] & Q_{r+q} \arrow[r] \arrow[d, "\gamma"] \arrow[ddl, dashed] & \cdots \arrow[r] \arrow[d, phantom, "\cdots"] & Q_r \arrow[d, "\gamma"] \\
Q_{p+q} \arrow[r, "\circlearrowright"] \arrow[d, "\beta"] & Q_{p+q-1} \arrow[r] \arrow[d, "\beta"] & \cdots \arrow[r] \arrow[d, phantom, "\cdots"] & Q_{q+2} \arrow[r, "\circlearrowright"] \arrow[d, "\beta"] & Q_{q+1} \arrow[r, "\circlearrowright"] \arrow[d, "\beta"] & Q_q \arrow[r] \arrow[d, "\beta"] & \cdots \arrow[r] & Q_0 \\
Q_p \arrow[r] \arrow[d, "\alpha"] & Q_{p-1} \arrow[r] & \cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 & & \\
Q_0 \arrow[d] & & & & & & & \\
k & & & & & & &
\end{tikzcd}$$

Examples

For each $n \in \mathbb{N} \cup \{0\}$, define the algebra B_n by

$$\frac{k\langle a_i, b_i, c_i\rangle_{0\leq i\leq n}}{\langle a_i b_i c_i + c_{i+1}a_{i+1}b_{i+1}, a_n b_n c_n, b_{i+1}c_{i+1}a_{i+1}, c_0 a_0, c_i c_{i+1}, b_{i+1}a_i\rangle_{0\leq i<n}}$$

Let $E(B_n)$ denote the Ext-algebra of B_n.

Theorem

For all $n \in \mathbb{N} \cup \{0\}$, B_n is a K_2 algebra.

Proof.

It is easy to show B_0 is K_2. The factor map $B_{n+1} \longrightarrow B_n$ shows B_n is K_2 as a B_{n+1}-module. Then a theorem of Cassidy and Shelton shows B_{n+1} is K_2 as an algebra. $\square$

Theorem

For all $n \in \mathbb{N} \cup \{0\}$, $E(B_n)$ has an A_∞ structure for which m_{n+3} is nonzero. Moreover this structure comes from the canonical quasi-isomorphism class of structures provided by Merkulov's construction.

The particular map is

$$m_{n+3} : E^{1,1} \otimes (E^{2,2})^{\otimes n} \otimes E^{1,1} \otimes E^{n+1,n+1} \to E^{2n+2,3n+3}$$

which has a nonzero value on the element

$$A_n \otimes (B_n A_{n-1}) \otimes \cdots \otimes (B_1 A_0) \otimes B_0 \otimes (C_0 C_1 \cdots C_n).$$

(Capitals for duals of generators)