Constructions of Algebras and a New Family of Quantum Projective 3-Spaces

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Outline

1. Introduction to algebras and some familiar examples.

2. Ore extensions, a new construction, and a family of examples.

3. Artin-Schelter regular algebras and noncommutative projective algebraic geometry.

Introduction

Throughout, let K be a field and $K^\times = K\backslash\{0\}$.

Definition. *A K-algebra A is a ring A with a linear action of K on A satisfying:*

$$k(ab) = (ka)b = a(kb)$$

for all $k \in K$ and $a, b \in A$.

• Equivalently, there is an injective ring homomorphism $i : K \to Z(A)$ with $i(1_K) = 1_A$ where

$$Z(A) = \{a \in A \mid ax = xa \text{ for all } x \in A\}$$

is the center of A.

Definition. *If A and B are K-algebras, a K-algebra homomorphism is a ring homomorphism $f : A \to B$ satisfying $f(ka) = kf(a)$ for all $k \in K$ and $a \in A$.*

• Algebras were historically first defined in the subjects of number theory, algebraic geometry, quaternions and generalizations.

Examples

1. The polynomial ring $K[x]$, or more generally, $K[x_0, \dots x_n]$,

   elements are finite sums over $\mathbb{N}^{n+1}$:

   $$f(x_0, \dots, x_n) = \sum_{i_0,\dots i_n} a_{i_0,\dots,i_n} x_0^{i_0} \cdots x_n^{i_n}$$

   where $a_{i_0,\dots,i_n} \in K$

2. Power series rings $K[[t]]$,

   elements: $f(t) = \sum_{i \geq 0} a_i t^i$ where $a_i \in K$

3. Matrix rings $M_n(K)$,

   elements: $A = (a_{ij})$ where $a_{ij} \in K$

4. Endomorphism rings, $\mathrm{End}_K(M)$ for a K-vector space M,

   elements: K- linear maps $T : M \to M$

Representation Theory of Algebras

Definition. *A (right) module M for the K-algebra A is a K-vector space M with a linear action of A on M satisfying*

$$m.(ab) = (m.a).b$$

for all $m \in M$ and $a \in A$.

• Equivalently a (right) module M is given by a K-algebra homomorphism $\rho : A \to \mathrm{End}_K(M)$.

• This point of view is very useful for studying A. We think of elements of the algebra as functions on M.

• If M is finite-dimensional over K, elements of a are represented by matrices acting on M.

Polynomial Algebras over K-algebras

- R, K-algebra
- x, (central) indeterminate
- Polynomials are finite formal sums $\sum_{i \geq 0} r_i x^i$ where $r_i \in R$.
- Let $R[x] = \{$ polynomials over R $\}$ and $p(x) = \sum_i p_i x^i$, $q(x) = \sum_i q_i x^i$:

$$p(x) + q(x) = \sum_i (p_i + q_i) x^i$$

$$p(x) \cdot q(x) = \sum_i \Big(\sum_{j+k=i} p_j q_k \Big) x^i$$

Definition. *Define the degree of $p(x)$ by*

$$\deg(p(x)) = n$$

if $p_n \neq 0$ and $p_k = 0$ for all $k > n$. Set $\deg(0) = -\infty$.

A Nice Property of Degree

- R a domain, (K-algebra),

$$\deg(p(x) \cdot q(x)) = \deg(p(x)) + \deg(q(x))$$

- $S = R[x]$ and let $S_n = \{rx^n \mid r \in R\}$ for $n \geq 0$.

Then

$$S = \oplus_{n \geq 0} S_n, \quad S_n \cdot S_m \subset S_{m+n}.$$

Definition. *Let A be a K-algebra.*

1. A is $\mathbb{N}$-graded if there exists a decomposition of K-vector spaces, $A = \oplus_{n \geq 0} A_n$ with

$$A_n \cdot A_m \subset A_{m+n}$$

for all $m, n \geq 0$.

2. A is connected if $A_0 = K$.

Definition. *Let A be an $\mathbb{N}$-graded K-algebra. A graded (right) A-module M is a right A-module M with a decomposition $M = \oplus_{n\in\mathbb{Z}} M_n$ as vector spaces such that*

$$M_n . A_m \subset M_{n+m}$$

for all $n \in \mathbb{Z}$, $m \geq 0$.

Definition. *Suppose $M = \oplus_{n\in\mathbb{Z}} M_n$ is a graded A-module with $\dim_K M_n < \infty$ for all $n \in \mathbb{Z}$. The Hilbert series of M is the (doubly-infinite) Laurent series*

$$H_M(t) = \sum_{n\in\mathbb{Z}} (\dim_K M_n) \cdot t^n.$$

ex. If $A = K[x_0, \ldots, x_n]$ (graded by total degree) then

$$H_A(t) = \sum_{m=0}^{\infty} \binom{n+m}{m} t^m = \frac{1}{(1-t)^{n+1}}.$$

Ore Extensions

- R, K-algebra

- construct an algebra (over R) with elements:

$$\sum_i x^i r_i,$$

(coefficients on right), $x^i \cdot x^j = x^{i+j}$, *but* x doesn't have to commute with coefficients

- want "degree" to behave nicely, insist

$$rx = x\sigma(r) + \delta(r),$$

for some linear maps $\sigma, \delta : R \to R$

- What kind of maps are σ and δ?

Let $r, s \in R$. On one hand,

$$(rs)x = x\sigma(rs) + \delta(rs).$$

On the other,

$$\begin{aligned}(rs)x = r(sx) &= r(x\sigma(s) + \delta(s))\\ &= (x\sigma(r) + \delta(r))\sigma(s) + r\delta(s)\\ &= x\sigma(r)\sigma(s) + \delta(r)\sigma(s) + r\delta(s).\end{aligned}$$

- $\sigma(rs) = \sigma(r)\sigma(s)$ and $\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$

- $\sigma \in \mathrm{End}(R)$ and δ is a "right σ-derivation"

- Let $S = R[x; \sigma, \delta]$.

Theorem. *S is a K-algebra.*

Theorem. *Let R be a K-algebra, $\sigma \in \mathrm{Aut}(R)$, δ a right σ-derivation. Let $S = R[x; \sigma, \delta]$.*

1. *Every element of S can be written uniquely as $\sum_{i \geq 0} x^i r_i$ for some $r_i \in R$.*

2. *If R is a domain then S is a domain.*

3. *If R is Noetherian then S is Noetherian.*

Examples

1. The polynomial algebra $K[x] = K[x; \mathrm{id}, 0]$.

2. The first Weyl algebra,

$$A_1(K) = K\langle x, y\rangle / \langle xy - yx - 1\rangle.$$

- Let $R = K[y]$ and consider the Ore extension $S = R[x; \mathrm{id}, \frac{d}{dy}]$.

- $S \cong A_1(K)$ as K-algebras.

Generalized Laurent Polynomial Algebras

• Generalizes the construction of generalized Weyl algebras introduced by V. Bavula.

Data:

(1) Let R be any K-algebra. Let $\sigma \in \mathrm{Aut}(R)$, and $q \in R$ a regular, normal element. So

$$qr = \tau(r)q$$

for some $\tau \in \mathrm{Aut}(R)$ and all $r \in R$.

(2) d, u indeterminates

Multiplication Rules:

(1) $rd = d\sigma(r)$ and $ur = \sigma(\tau(r))u$ for all $r \in R$

(2) $du = q$ and $ud = \sigma(q)$

• Let $S = R[d, u; \sigma, q]$ be the set of all formal sums:

$$\sum_{i \geq 0} a_i d^i + \sum_{j > 0} b_j u^j$$

for $a_i, b_j \in R$ with multiplication given by the above rules.

Theorem (CGSh). *S is a K-algebra and:*

1. Every element of S can be written uniquely as

$$\sum_{i \geq 0} a_i d^i + \sum_{j > 0} b_j u^j$$

for some $a_i, b_j \in R$.

2. If R is a domain then S is a domain.

3. If R is Noetherian then S is Noetherian.

• S is a subalgebra of a skew Laurent polynomial algebra hence the name generalized Laurent polynomial algebra

Examples

1. The quantum plane

$$K_\lambda[x,y] = K\langle x,y\rangle/\langle xy-\lambda yx\rangle$$

for $\lambda \in K^\times$.

2. $\mathcal{U}(sl_2)$, enveloping algebra of Lie algebra sl_2.

A New Family of Examples

- $r, \alpha \in K^{\times}$

- Let $A(r)$ be the "down-up" algebra $K\langle x, y\rangle$ factored by

$$\langle x^2y-(r+r^{-1})xyx+yx^2, y^2x-(r+r^{-1})yxy+xy^2\rangle.$$

- Let $C(r,\alpha) = A(r)[d,u;\sigma,q]$ where

$$\sigma : x \mapsto y,\ \ \sigma : y \mapsto \alpha x$$

and $q = (xy - ryx)$.

- q is not central in $A(r)$.

Theorem (CGSh). *$C(r,\alpha)$ is isomorphic to the algebra $K\langle x, y, d, u\rangle$ factored by the ideal generated by the six quadratic elements*

$$xd - dy, \quad yd - \alpha dx, \quad ux - ryu, \quad uy - \alpha sxu,$$

$$du - (xy - ryx), \qquad ud + \alpha r(xy - syx).$$

Noncommutative Algebraic Geometry

Definition (AS). *An $\mathbb{N}$-graded, connected, K-algebra A is AS-regular of dimension d if:*

1. *A has global dimension d.*

2. *A has finite GK-dimension.*

3. *A satisfies the Gorenstein condition:*

$$\mathit{Ext}^n(K, A) = \delta_{n,d} K.$$

K denotes the trivial A-module.

Remark. *If A is AS-regular and commutative then A is a polynomial algebra.*

• think of AS-regular algebras as noncommutative deformations or quantizations of polynomial algebras

Classifications

Dimension 2: Up to isomorphism: quantum planes $K_\lambda[x, y]$ and $K\langle x, y\rangle/\langle yx - xy - x^2\rangle$.

Dimension 3: Artin, Schelter, Tate, Van den Bergh classified "generated in degree 1". Stephenson classified "not generated in degree 1".

• 2 types in dimension 3 (generated in degree 1):

1. 3 generators, 3 quadratic relations (like polynomial algebra)

2. 2 generators, 2 cubic relations (like $A(r)$ on previous slide)

Technique: Associate certain graded modules playing the roles of points, lines, etc....

Dimension 4: Not classified, many examples have been studied.

Definition. *A quantum* $\mathbb{P}^3$ *is an AS-regular algebra* A *of dimension 4 whose Hilbert series is* $\frac{1}{(1-t)^4}$*. In particular,* A *can be presented as an algebra on 4 linear generators and* $6 = \binom{4}{2}$ *quadratic relations.*

- $R = K[x_0, x_1, x_2, x_3]$ is the homogeneous coordinate ring of $\mathbb{P}^3$, and $H_R(t) = \frac{1}{(1-t)^4}$.

Theorem (CGSh). *For any* $r, \alpha \in K^\times$, $C(r, \alpha)$ *is a quantum* $\mathbb{P}^3$*.*

Geometry

Definition. *Let A be an AS-regular algebra of dimension n. A point module P is a graded, cyclic module which is generated in degree 0, and $H_P(t) = 1/(1-t)$.*

• If $R = K[x_0, \ldots, x_n]$, point modules correspond to points in $\mathbb{P}^n$.

• If $n \leq 4$ there is a projective scheme Γ that parametrizes the point modules. Furthermore Γ is the graph of an automorphism s. We call (Γ, s) the point scheme of A.

A link between Algebra and Geometry

Theorem (ATV). *Let A be an AS-regular algebra of dimension 3. A is a finitely generated module over its center if and only if s has finite order.*

Theorem (CGSh). *Suppose that r is not a root of unity and α is a root of unity.*

1. *$C(r, \alpha)$ is not a finitely generated module over its center.*
2. *The point scheme is finite (4 points, not reduced) and the associated automorphism has finite order.*

Theorem (StV, CGSh). *There exist quantum $\mathbb{P}^3$'s where s has finite order and yet the algebra is not a finitely generated module over its center.*

More Geometry: Fat Point Modules

Definition. *Let A be a quantum $\mathbb{P}^3$ and $m \in \mathbb{Z}_{>1}$. A fat point module of multiplicity m is a graded, cyclic, A-module that is generated in degree 0, has Hilbert series $m/(1-t)$, and is critical with respect to GK-dimension.*

- The automorphism s naturally extends to a function s on the set of fat point modules.

Theorem (GSh). *The generic member of the Stephenson-Vancliff examples has a one-parameter family of fat points of multiplicity 2 upon which s has infinite order.*

Questions:

1. What are the fat point modules of $C(r, \alpha)$?

2. How does s act on the fat point modules?

Definition. *Let A be a connected, $\mathbb{N}$-graded K-algebra with center Z. If M is a graded A-module we say the center acts trivially on M if $MZ_{>1} = 0$. Otherwise we say the center acts nontrivially on M.*

Theorem. *Assume r is not a root of unity and α is a root of unity. Let F be a fat point module for $C(r, \alpha)$. Then the multiplicity of F is 2.*

1. The family of fat point modules for which the center acts trivially is parametrized by $K^\times$ and s acts by multiplication by r. In particular s has infinite order when the center acts trivially.

2. The family of fat point modules for which the center acts non-trivially is parametrized by $K^\times$ and s acts by multiplication by $\alpha^{-1/2}$. In particular s has finite order when the center acts nontrivially.

Proofs

(1) Use the noncommutative algebraic geometry of some cubic AS-regular algebras of dimension 3.

(2) Use the ATV theorem in a nice way.

Conjecture. Let A be a quantum $\mathbb{P}^3$. Suppose s has finite order on the scheme of point modules and finite order on all fat points. Then A is a finitely generated module over its center.

Questions:

1. What happens when r is a root of unity or α isn't a root of unity?
2. If A is finitely generated over its center, does s have finite order on the point scheme and finite order on the fat points?
3. Are the fat point modules (of fixed multiplicity) parametrized by a (quasi-projective) scheme?

More examples needed! (Undergraduate research?)