The Cohomology of the "Group of Loops"

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Outline

1. The Group of Loops
2. Cohomology
3. Koszulity
4. Current Work

The Group of Loops

- n: fixed positive integer

- $[x, y]$: commutator (group or algebra, depending on context), or Lie bracket

The Group of Loops

- Generators: $\{a_{ij} \mid 1 \leq i \neq j \leq n\}$
- Relations: for distinct indices i, j, k, l,

$$[a_{ij}, a_{kj}],$$

$$[a_{ij}, a_{ik}a_{jk}],$$

$$[a_{ij}, a_{kl}]$$

- G: the group with the above generators and relations
- Why is G interesting?

Combinatorial Group Theory

- $F := \langle x_1, \ldots, x_n \rangle$, free group
- Define automorphisms of F:

$$a_{ij}(x_i) = x_j x_i x_j^{-1},$$

$$a_{ij}(x_k) = x_k, \text{ for } k \neq i$$

- $P\Sigma$: subgroup of $\mathrm{Aut}(F)$ generated by $a_{ij}, 1 \leq i \neq j \leq n$

Theorem (McCool, 1986)

The group $P\Sigma$ is isomorphic to G.

- Call G "the McCool group"

Topology

- $\mathcal{L}$: collection of n unknotted, unlinked circles in $\mathbb{R}^3$
- M: the **motion group** of $\mathcal{L}$
- Roughly, elements of M are finite length movies consisting of n circles moving about in space, they take $\mathcal{L}$ to itself (but the circles can be permuted at the end).
- PM: **pure motion group**, the subgroup of M consisting of movies which return the circles to their original positions
- $\alpha_{ij} \in PM$: the movie "pull C_i through C_j"

Topology

- Basepoint $e \notin \mathcal{L}$
- x_i: loop based at e linking C_i once
- Identify $\pi_1(\mathbb{R}^3 - \mathcal{L}, e)$ with the free group F
- Dahm: There exists a homomorphism
  $\phi : M \longrightarrow \mathrm{Aut}(\pi_1(\mathbb{R}^3 - \mathcal{L}, e))$
- Goldsmith, Brendle-Hatcher: ϕ is an embedding, $\phi : \alpha_{ij} \mapsto a_{ij}$,
  $\phi : PM \cong G$

Cohomology of G

- k: field, (or commutative ring)
- $A := H^*(G,k)$: cohomology algebra of G
- Brownstein and Lee (1993): The algebra A is a quotient of the exterior algebra E on generators $e_{ij}, 1 \leq i \neq j \leq n$. For all distinct i, j, k, the following relations hold in A:

$$e_{ij} e_{ji},$$

$$(e_{ki} - e_{ji})(e_{kj} - e_{ij}).$$

Theorem (Jensen, McCammond, Meier, 2006)

The above relations give a presentation of A as a quotient of the exterior algebra E. Consequently, the Hilbert series of A is $(1+nt)^{n-1}$.

Koszulity

- G^+: generators, $a_{ij}, 1 \leq i < j \leq n$; relations, same as for G
- Call G^+ the "upper McCool group"
- Motivated by rational homotopy theory...

Theorem (Cohen, Pruidze, 2008)

The cohomology $H^*(G^+, k)$ is a Koszul algebra.

- Cohen and Pruidze: Is the algebra A Koszul?

Current Work

- $B := A^!$: quadratic dual algebra of A

- Presentation of B: generators $x_{ij}, 1 \leq i \neq j \leq n$; relations (for distinct i, j, k, l),

$$[x_{ij}, x_{kj}],$$

$$[x_{ij}, x_{ik} + x_{jk}],$$

$$[x_{ij}, x_{kl}]$$

- Try to show B is Koszul

Small cases

- $n = 3$: B (or A) is a PBW algebra, hence Koszul
- $n = 4$: numerical Koszulity $\implies$ Koszulity, (but no proof of numerical Koszulity)
- $n = 4, 5$: Bergman demonstrates Koszulity up to some fixed (small) degree

Consequence of B's Koszulity

Rational homotopy theory

- Sullivan: a connected finite type CW-complex X is *formal* if the rational homotopy type of X is determined by the cohomology algebra $H^*(X, \mathbb{Q})$
- 1-formal spaces: spheres, simply connected Eilenberg-MacLane spaces, complements of certain complex hyperplane arrangements
- Quillen: a finitely presented group is *1-formal* if its Malcev Lie algebra is quadratic
- Berceanu, Papadima: The McCool group is 1-formal
- Papadima, Suciu: B Koszul $\implies$ $X := K(G, 1)$ is formal

Consequence of B's Koszulity

Welded Braids

- Braid permutation group: $BP = G \rtimes \Sigma$, where Σ is the permutation group on n letters
- BP: studied by Fenn, Rourke, Remanyi in the context of *welded braids*
- B Koszul $\implies H_B(t) = \dfrac{1}{(1-nt)^{n-1}}$
- This determines the maximal number of *linearly independent finite type weight systems for welded braids* of degree j, for all j

A reduction in the general case

- Working in B
- Let $X_j = \sum_{i \neq j} x_{ij}$
- S: subalgebra of B generated by $X_1, \ldots, X_n$

Proposition

S is a free algebra and a normal ($BS_+ = S_+B$) subalgebra.

- I: the two-sided ideal of B generated by $X_1, \ldots, X_n$
- Standard spectral sequence argument shows:

  Koszulity of $\overline{B} := B/I \implies$ Koszulity of B

A Lie algebra

Motivated by the theory of quantum groups, Bartholdi, Enriquez, Etingof, Rains define:

- $\mathfrak{qtr}_n$: n-th quasitriangular Lie algebra
- Generators: r_{ij}, $1 \leq i \neq j \leq n$
- Relations: for distinct i, j, k, l

$$[r_{ij}, r_{kl}],$$

$$[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}]$$

(classical Yang-Baxter relation)

- $\mathfrak{qtr}_n$: the free Lie algebra on the r_{ij} modulo the relations
- U: $\mathcal{U}(\mathfrak{qtr}_n)$, the universal enveloping algebra

Theorem (Bartholdi, Enriquez, Etingof, Rains)

U is Koszul.

Their method of proof is interesting.

- Write $r_{ij} = t_{ij} + \rho_{ij}$; t's are symmetric, ρ's skewsymmetric
- Rewrite relations in the new variables
- Compute the corresponding presentation of the quadratic dual, Q
- Filter the algebra Q; the relations become inhomogeneous
- Consider the graded versions of the relations and form the corresponding graded algebra Q^0
- There's a surjection $Q^0 \to \mathrm{gr}Q$
- Q^0 is PBW and the map $Q^0 \to \mathrm{gr}Q$ is an isomorphism
- Finally, $\mathrm{gr}Q$ Koszul $\implies$ Q Koszul $\implies$ U Koszul

New idea

- Observe, there's a natural surjection $U \to B$
- Idea, play the above game on B (or $\overline{B}$)
- Unfortunately, the associated graded is not Koszul
- However, it does appear to be $\mathcal{K}_2$ (generalization of Koszul due to Cassidy and Shelton)

Question

Let B be a quadratic algebra, F a filtration and $\mathrm{gr}B$ the associated graded algebra. If $\mathrm{gr}B$ is $\mathcal{K}_2$ and $q(\mathrm{gr}B)$ is Koszul, must B be Koszul?