An Introduction to Koszul Algebras

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Outline

1) Graded algebras, quadratic algebras

2) Quadratic duals, Ext

3) Koszul algebras, examples

Noncommutative polynomials

Let $\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle$ be the set of noncommutative polynomials in the variables $x_1, x_2, \ldots, x_n$ with complex number coefficients.

$$(5+2i)x_3^4 - x_1x_2 \neq (5+2i)x_3^4 - x_2x_1$$

In this set you can add and multiply just like with "regular" polynomials, except you can't do as much simplifying.

$$(x_1+x_2)(x_1+x_2) = x_1^2 + x_1x_2 + x_2x_1 + x_2^2 \neq x_1^2 + 2x_1x_2 + x_2^2$$

$\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle$ is an *algebra* because we can add, multiply, and multiply by scalars in the "usual" way.

$\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle$ is called a *free algebra*.

A noncommutative game

You make up rules in $\mathbb{C}\langle x_1, x_2, \ldots, x_n \rangle$, called *relations*, which tell you when certain polynomials are equal.

Example

In $\mathbb{C}\langle x_1, x_2 \rangle$, say that $x_1 x_2 = x_2 x_1$. We'll get the ordinary commutative polynomials in two variables.

Example

In $\mathbb{C}\langle x_1, x_2 \rangle$, say that $x_1^2 = 0$, $x_2^2 = 0$ and $x_1 x_2 = -x_2 x_1$. We'll get something interesting as we'll soon see.

Relations

You might have noticed that the relations in the last two examples preserved degree. We didn't have a relation like $x_1^2 = x_1 + x_2$. You can do that, but we won't in this talk.

Also note that the relations in the examples were degree 2, that is *quadratic*. You can have relations of degree greater than 2, and that's really interesting! We'll mainly stick to quadratic relations in this talk.

Finally, notice that in the examples if we multiply two polynomials, f and g, then $\deg(fg) = \deg(f) + \deg(g)$.

Definition of graded algebra

Let $A = \mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle / \langle R\rangle$, where R is the set of relations.

$$A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \cdots ,$$

where A_k is the finite dimensional vector space spanned by the homogeneous polynomials in A of degree k.
Note that $A_k \cdot A_l \subseteq A_{k+l}$.
We say that A is a *graded algebra*.

Problem

Find the dimension of A_k as a function of k.

The *Hilbert series* of A is the power series

$$H_A(t) = \sum_{k=0}^{\infty} \dim(A_k) t^k .$$

Example, the free algebra

Let $A = \mathbb{C}\langle x_1, \dots, x_n \rangle$.

Then A_k is the vector space spanned by all noncommutative homogeneous polynomials of degree k.

The number of monomials of degree k can be counted by thinking of k boxes, and since there are n choices for what to put in each box, $\dim A_k = n^k$.

Therefore

$$H_{\mathbb{C}\langle x_1, \dots, x_n \rangle}(t) = 1 + nt + n^2 t^2 + n^3 t^3 + \cdots = \frac{1}{1 - nt}.$$

Example, the commutative polynomial algebra

Let $A = \mathbb{C}\langle x_1, \ldots, x_n \rangle / \langle x_i x_j = x_j x_i \rangle$.
Then A_k is the vector space of all commutative homogenous polynomials of degree k.

$$\dim A_k = \binom{k+n-1}{n-1}; H_A(t) = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} t^k = \frac{1}{(1-t)^n}$$

Example

$n = 4,\ k = 7$; variables: x_1, x_2, x_3, x_4; monomial of degree 7: $x_1^{d_1} x_2^{d_2} x_3^{d_3} x_4^{d_4}$, $d_1 + d_2 + d_3 + d_4 = 7$.

$$\boxed{x_1}\boxed{x_1}\boxed{\,|\,}\boxed{x_2}\boxed{x_2}\boxed{x_2}\boxed{\,|\,}\boxed{x_3}\boxed{\,|\,}\boxed{x_4} = x_1^2 x_2^3 x_3 x_4$$

Example, the exterior algebra

Let $A = \mathbb{C}\langle x_1, \ldots, x_n\rangle / \langle x_i^2 = 0, x_i x_j = -x_j x_i \text{ for } i < j\rangle$.

Because of the relations, any monomial of degree k can be put in the form

$$x_{i_1} \cdots x_{i_k}, \; i_1 < i_2 < \cdots < i_k.$$

It can be shown that these are linearly independent.

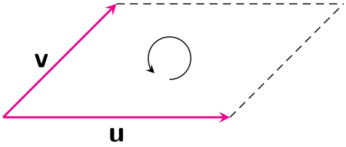
Therefore $\dim A_k = \binom{n}{k}$.

$$H_A(t) = 1 + nt + \binom{n}{2} t^2 + \cdots + \binom{n}{n-2} t^{n-2} + nt^{n-1} + t^n = (1+t)^n.$$

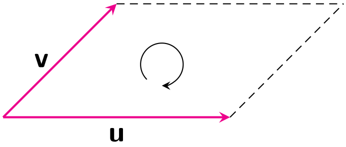
Geometry and the exterior algebra

Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^2$.

$\mathbf{u} \wedge \mathbf{v} \quad =$



$= - \Big\{$



$\Big\} \quad = \quad -(\mathbf{v} \wedge \mathbf{u})$

We can think of *the wedge product* as a geometrically defined multiplication.

Geometry and the exterior algebra

Let $\mathbf{e_1}, \mathbf{e_2}$ denote the standard basis vectors for $\mathbb{R}^2$.

Write $\mathbf{u} = a\mathbf{e_1} + b\mathbf{e_2}, \mathbf{v} = c\mathbf{e_1} + d\mathbf{e_2}$ for some $a, b, c, d \in \mathbb{R}$.

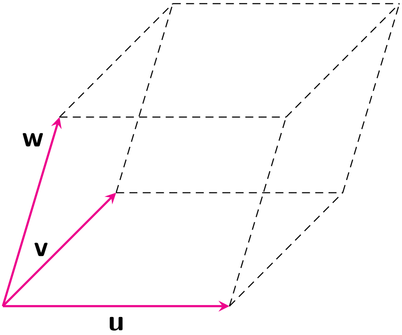
Then

$$\begin{aligned}\mathbf{u} \wedge \mathbf{v} &= (a\mathbf{e_1} + b\mathbf{e_2}) \wedge (c\mathbf{e_1} + d\mathbf{e_2}) \\ &= ac\mathbf{e_1} \wedge \mathbf{e_1} + ad\mathbf{e_1} \wedge \mathbf{e_2} + bc\mathbf{e_2} \wedge \mathbf{e_1} + bd\mathbf{e_2} \wedge \mathbf{e_2} \\ &= (ad - bc)\mathbf{e_1} \wedge \mathbf{e_2}.\end{aligned}$$

Recall that the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is $\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ad - bc|$.

Geometry and the exterior algebra

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$.

$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} =$ 

Writing $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in the standard basis, $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, of $\mathbb{R}^3$ yields

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = (\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}))\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}.$$

Recall that the volume of the parallelopiped is given by the absolute value of the *triple product* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Invariants

There is a notion of when two graded algebras should be considered the same. It is called *isomorphism*.

Problem

Given two algebras A and B, decide if they are isomorphic.

One way to do this is to find an invariant that is computable and can be used to tell if A and B are not isomorphic.

Two such invariants are *the quadratic dual* and *the Ext algebra*.

The quadratic dual

Let $A = \mathbb{C}\langle x_1, \ldots, x_n\rangle / \langle R\rangle$, where R is some set of quadratic relations.

Let $y_1, \ldots, y_n$ be dual variables to $x_1, \ldots, x_n$, that is,

$$y_i(x_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} .$$

Define a new algebra $A^! = \mathbb{C}\langle y_1, \ldots, y_n\rangle / \langle R^\perp\rangle$, where $R^\perp$ is the set of all homogeneous degree 2 polynomials in the y_i that are zero on all the relations in R.

Example, quadratic dual of commutative polynomials in two variables

Let $A = \mathbb{C}\langle x_1, x_2\rangle / \langle x_1 x_2 = x_2 x_1\rangle$.

Let y_1, y_2 be dual variables to x_1, x_2.

$$y_1^2(x_1x_2 - x_2x_1) = y_1(x_1)y_1(x_2) - y_1(x_2)y_1(x_1) = 1 \cdot 0 - 0 \cdot 1 = 0$$

Similarly, $y_2^2(x_1x_2 - x_2x_1) = 0$ and $(y_1y_2 + y_2y_1)(x_1x_2 - x_2x_1) = 0$. Any other degree 2 polynomial in y_1 and y_2 which is zero on $x_1x_2 - x_2x_1$ is a linear combination of $y_1^2, y_2^2, y_1y_2 + y_2y_1$.

Therefore

$$\left[\mathbb{C}\langle x_1, x_2\rangle / \langle x_1x_2 = x_2x_1\rangle\right]^! = \mathbb{C}\langle y_1, y_2\rangle / \langle y_1^2 = 0, y_2^2, y_1y_2 = -y_2y_1\rangle.$$

The Ext algebra

Let $A = \mathbb{C}\langle x_1, \ldots, x_n \rangle / \langle R \rangle$. Suppose that $R = \{r_1, \ldots, r_m\}$.

We build a *minimal free resolution* of $\mathbb{C}$.

$$\cdots \xrightarrow{M_3} A^{b_2} \xrightarrow{M_2} A^{b_1} \xrightarrow{M_1 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} A \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0 \,,$$

where the M_i are matrices whose entries are polynomials from A, and $\epsilon(\text{polynomial}) = \text{constant term}$.

Let $E^k = \mathrm{Hom}_A(A^{b_k}, \mathbb{C})$. Then $E^k \cong \mathbb{C}^{b_k}$.

The *Ext algebra* is the graded algebra $E = \mathbb{C} \oplus E_1 \oplus E_2 \oplus \cdots$.

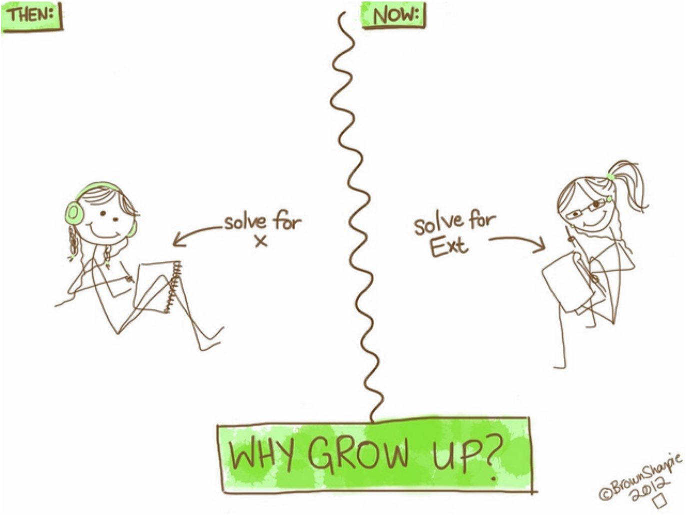
Multiplying in the Ext algebra

Let $\alpha \in E^i = \mathrm{Hom}_A(A^{b_i}, \mathbb{C})$, $\beta \in E^j = \mathrm{Hom}_A(A^{b_j}, \mathbb{C})$.

$$\begin{tikzcd}
A^{b_{i+j}} \arrow[r, dotted, no head] \arrow[d, dashed, "\beta_i"] & A^{b_{j+1}} \arrow[r, "M_{j+1}"] \arrow[d, dashed, "\beta_1"] & A^{b_j} \arrow[d, dashed, "\beta_0"] \arrow[dr, "\beta"] & \\
A^{b_i} \arrow[r, dotted, no head] \arrow[d, "\alpha"] & A^{b_1} \arrow[r, "M_1"'] & A \arrow[r, "\epsilon"'] & \mathbb{C} \\
\mathbb{C} & & &
\end{tikzcd}$$

Define $\alpha \star \beta \in E^{i+j} = \mathrm{Hom}_A(A^{b_{i+j}}, \mathbb{C})$ by $\alpha \star \beta := \alpha \circ \beta_i$.

Solving for Ext?



Definition of Koszul algebra

Let

$$\cdots \xrightarrow{M_3} A^{b_2} \xrightarrow{M_2} A^{b_1} \xrightarrow{M_1} A \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0$$

be a minimal free resolution of $\mathbb{C}$.

Definition (Stewart Priddy)

The graded algebra A is *Koszul* if all the matrix entries of all the M_i are linear polynomials from A.

Koszul Algebras (history)

Stewart Priddy introduced the notion of *Koszul algebra* in 1970.
He was interested in computing the cohomology of topological spaces, specifically, he was studying the (stable) homotopy groups of spheres.
Priddy called them Koszul algebras in honor of Jean-Louis Koszul who had studied related notions for enveloping algebras of Lie algebras in the 1950s.

Koszul algebras and quadratic algebras

"Koszul" $\implies$ "quadratic", but the converse is false.

Koszul algebras arise in many areas: topology, algebraic geometry (commutative and non-commutative), representation theory, number theory, combinatorics.

Theorem (Priddy)

Let A be a quadratic algebra. Then A is Koszul if and only if $E(A) \cong A^!$.

Corollary

Let A be a Koszul algebra. Then $H_A(t)H_{A^!}(-t) = 1$.

So Koszul algebras are algebras in which: "Ext is easy to solve for" and we get information on Hilbert series.

Example, commutative polynomials on two variables

Let $A = \mathbb{C}\langle x_1, x_2\rangle / \langle x_1 x_2 = x_2 x_1\rangle$.

To prove A is Koszul we have to resolve $\mathbb{C}$ and see what the matrices look like. Using the fact that we have a nice basis for A, namely $\{x_1^i x_2^j \mid i, j \geq 0\}$, it is not hard to show

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} -x_2 & x_1 \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} A \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0$$

is the minimal free resolution of $\mathbb{C}$.

Therefore A is Koszul and
$E(A) \cong \mathbb{C}\langle y_1, y_2\rangle / \langle y_1^2 = 0, y_2^2 = 0, y_1 y_2 = -y_2 y_1\rangle$.

Example, computing a Yoneda product

Let's check that $y_1 \star y_1 = 0$.

$$
\begin{tikzcd}
A \arrow[r, "\begin{pmatrix}-x_2 & x_1\end{pmatrix}"] \arrow[d, dashed, "\begin{pmatrix}0 & -1\end{pmatrix}"'] & A \oplus A \arrow[d, dashed, "\begin{pmatrix}1\\0\end{pmatrix}"'] \arrow[dr, "y_1"] & \\
A \oplus A \arrow[r, "\begin{pmatrix}x_1\\x_2\end{pmatrix}"'] \arrow[d, "y_1"'] & A \arrow[r, "\epsilon"'] & \mathbb{C} \\
\mathbb{C} & &
\end{tikzcd}
$$

Since the composition $\begin{pmatrix}0 & -1\end{pmatrix} y_1 = 0$, we get $y_1 \star y_1 = 0$.

Hyperplane Arrangements

A *hyperplane* is the solution set of $a_1x_1 + \cdots a_nx_n = 0$ in $\mathbb{C}^n$.

We will identify hyperplanes with their equations.

A *hyperplane arrangement* is a finite set, $\mathcal{H} = \{H_1, \ldots, H_m\}$, of hyperplanes in $\mathbb{C}^n$.

Definition

The *braid arrangement* $\mathcal{B}_n$ is

$$\{x_i - x_j = 0 \mid 1 \leq i < j \leq n\}.$$

So there are $\binom{n}{2}$ hyperplanes.

The $\mathcal{B}_3$ hyperplane arrangement in $\mathbb{R}^3$

$$\mathcal{B}_3 = \{x_1 - x_2, x_1 - x_3, x_2 - x_3\}$$

Orlik-Solomon algebra

The complement of a complex arrangement is, topologically speaking, very interesting.

For example, the fundamental group of the complement of the braid arrangement is the famous pure braid group.

The *Orlik-Solomon algebra* is $\mathbb{C}\langle x_1, \ldots, x_n\rangle / \langle R\rangle$, where R is the set of relations

$$x_i^2 = 0 \qquad\qquad i = 1, \ldots n$$

$$x_i x_j = -x_j x_i \qquad\qquad 1 \le i < j \le n$$

$$(x_{i_2} - x_{i_1})(x_{i_3} - x_{i_1}) \cdots (x_{i_p} - x_{i_1}) = 0,$$

whenever $H_{i_1}, \ldots, H_{i_p}$ are dependent.

Koszulity

Theorem (Kohno, Shelton-Yuzvinsky)

The Orlik-Solomon algebra of the braid arrangement is Koszul.

The braid arrangement fits into the context of Coxeter groups and finite reflection groups. It is related to the A_n series.

My colleague, Andrew Conner, and I are writing a paper on algebras related to the braid arrangement and the Koszul property.

Thank you for listening!