Some non-Koszul Algebras I Met On Sabbatical

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Outline

1) Graded algebras, Ext, and Koszul algebras

2) The Algebras

Graded algebras

Let k denote a field - you can think of $k = \mathbb{Q}, \mathbb{C}$, if you'd like. Most objects in the talk will be (at least) vector spaces over k.

Let A denote a unital associative algebra over k.

We say A is *connected, graded, locally finite* if:

1) as a vector space

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots = \bigoplus_{i \geq 0} A_i,$$

where each A_i is a finite-dimensional vector space over k,

2) $A_i A_j \subseteq A_{i+j}$, $\forall i, j \geq 0$,
3) $A_0 = k$.

Graded algebras

From now on, A will denote a connected, graded, locally finite, unital associative algebra over k - we'll say A is a *k-algebra* for short.

Each $a \in A$ can be written uniquely as

$$a = a_0 + a_1 + \cdots + a_k, \quad a_i \in A_i.$$

We say a_i is *homogeneous of degree i*.

The *Hilbert series of A* is

$$H_A(t) := \sum_{i=0}^{\infty} \dim_k(A_i) t^i.$$

It's generally a challenging problem to compute $H_A(t)$; equivalently, determine all $\dim_k(A_i)$.

Example 1, the tensor algebra

Let V be an n-dimensional vector space over k.

The *tensor algebra on V* is defined as follows. As a vector space,

$$T(V) := k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots = \bigoplus_{i \geq 0} V^{\otimes i}.$$

Multiplication is given by

$$(x_1 \otimes \cdots \otimes x_i) \cdot (y_1 \otimes \cdots \otimes y_j) = x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j.$$

Since $\dim_k V^{\otimes i} = n^i$, we get

$$H_{T(V)}(t) = 1 + nt + n^2 t^2 + \cdots = \frac{1}{1 - nt}.$$

Example 2, the polynomial algebra

Let V be an n-dimensional vector space over k.

Set

$$S(V) := T(V)/\langle x \otimes y - y \otimes x \mid x, y \in V \rangle.$$

Then $S(V)$ is the k-algebra of polynomials in n variables, say $x_1, \dots, x_n$.

The dimension of the ith graded piece is

$$\dim_k S(V)_i = \#\left\{ x_1^{d_1} \cdots x_n^{d_n} \;\middle|\; \sum_{j=1}^{n} d_j = i \right\} = \binom{i+n-1}{n-1}.$$

Therefore $H_{S(V)}(t) = \dfrac{1}{(1-t)^n}$.

Example 3, the exterior algebra

Let V be an n-dimensional vector space over k.

Set

$$\Lambda(V) := T(V)/\langle x \otimes x \mid x \in V \rangle;$$

we'll use $\wedge$ for the multiplication operator on $\Lambda(V)$.

Note that $x \wedge y = -(y \wedge x)$ for all $x, y \in V$.

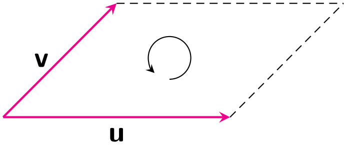
It follows that $\dim_k \Lambda(V)_i = \binom{n}{i}$, so

$$H_{\Lambda(V)}(t) = 1 + nt + \binom{n}{2} t^2 + \cdots + nt^{n-1} + t^n = (1+t)^n.$$

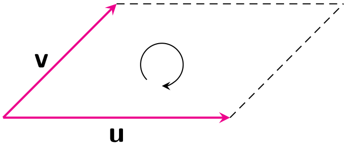
Geometry and the exterior algebra

Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^2$.

$\mathbf{u} \wedge \mathbf{v} =$



$= -\Big\{$



$\Big\} = -(\mathbf{v} \wedge \mathbf{u})$

We can think of *the wedge product* as a geometrically defined multiplication.

Geometry and the exterior algebra

Let $\mathbf{e_1}, \mathbf{e_2}$ denote the standard basis vectors for $\mathbb{R}^2$.

Write $\mathbf{u} = a\mathbf{e_1} + b\mathbf{e_2}, \mathbf{v} = c\mathbf{e_1} + d\mathbf{e_2}$ for some $a, b, c, d \in \mathbb{R}$.

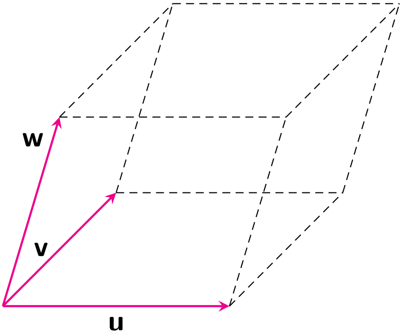
Then

$$\begin{aligned}\mathbf{u} \wedge \mathbf{v} &= (a\mathbf{e_1} + b\mathbf{e_2}) \wedge (c\mathbf{e_1} + d\mathbf{e_2}) \\ &= ac\mathbf{e_1} \wedge \mathbf{e_1} + ad\mathbf{e_1} \wedge \mathbf{e_2} + bc\mathbf{e_2} \wedge \mathbf{e_1} + bd\mathbf{e_2} \wedge \mathbf{e_2} \\ &= (ad - bc)\mathbf{e_1} \wedge \mathbf{e_2}.\end{aligned}$$

Recall that the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is $\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ad - bc|$.

Geometry and the exterior algebra

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$.

$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} =$ 

Writing $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in the standard basis, $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, of $\mathbb{R}^3$ yields

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = (\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}))\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}.$$

Recall that the volume of the parallelopiped is given by the absolute value of the *triple product* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

The Ext algebra (as a graded vector space)

We are going to define the *Ext algebra*

$$E = \operatorname{Ext}_A(k,k) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_A(k,k) = \bigoplus_{i\geq 0} E^i.$$

Let

$$\cdots \xrightarrow{\quad f_2 \quad} P_2 \xrightarrow{\quad f_1 \quad} P_1 \xrightarrow{\quad f_0 \quad} P_0 \xrightarrow{\quad \epsilon \quad} k \longrightarrow 0$$

be a resolution of ${}_A k$ by graded projective left A-modules.
Now apply the graded dual functor, $\operatorname{Hom}_A({}_-, k)$.

$$0 \longrightarrow \operatorname{Hom}_A(P_0,k) \xrightarrow{f_0^*} \operatorname{Hom}_A(P_1,k) \xrightarrow{f_1^*} \operatorname{Hom}_A(P_2,k) \xrightarrow{f_2^*} \cdots$$

Then define $\operatorname{Ext}^i_A(k,k) := \ker f_i^* / \operatorname{im}\, f_{i-1}^*$.

The Ext algebra (as a graded algebra)

Theorem

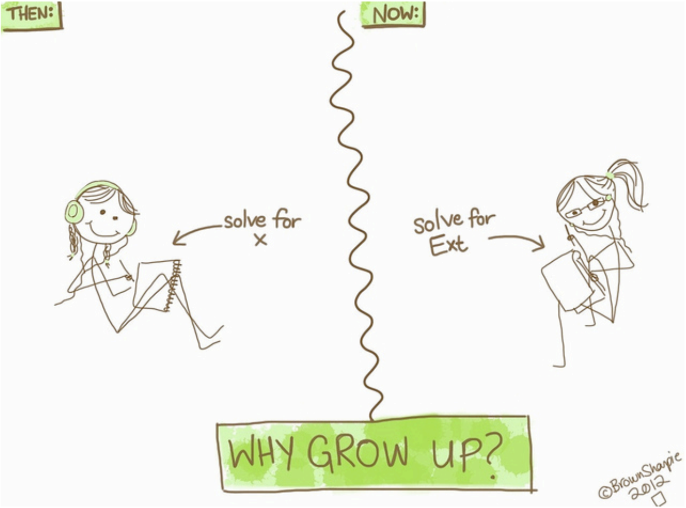
The left A-module ${}_A k$ has a unique (up to isomorphism) minimal resolution.

Here *minimal* means $f_i^* = 0$ for all i.
So if $P_\bullet \to k$ is minimal, then $E^i = \mathrm{Hom}_A(P_i, k)$.
We want to define a natural multiplication, $\star$, on the graded vector space E, so let $\alpha \in E^i = \mathrm{Hom}_A(P_i, k)$, $\beta \in E^j = \mathrm{Hom}_A(P_j, k)$.

$$
\begin{tikzcd}
P_{i+j} \arrow[r, dotted, no head] \arrow[d, dashed, "\beta_i"] & P_{j+1} \arrow[r, "f_j"] \arrow[d, dashed, "\beta_1"] & P_j \arrow[d, dashed, "\beta_0"] \arrow[dr, "\beta"] & \\
P_i \arrow[r, dotted, no head] \arrow[d, "\alpha"] & P_1 \arrow[r, "f_0"'] & P_0 \arrow[r, "\epsilon"'] & k \\
k & & &
\end{tikzcd}
$$

Define $\alpha \star \beta \in E^{i+j} = \mathrm{Hom}_A(P_{i+j}, k)$ by $\alpha \star \beta := \alpha \circ \beta_i$.

Solving for Ext?



Quadratic algebras and their duals

A k-algebra A is quadratic if it can be presented as $A = T(V)/\langle R \rangle$ where $R \subseteq V \otimes V$ is a subspace.

Note that all of the examples we gave above are quadratic. Quadratic algebras are ubiquitous in mathematics.

Every quadratic algebra has a natural *quadratic dual*:

$$A^! := T(V^*)/\langle R^\perp \rangle,$$

where $R^\perp = \{h \in V^* \otimes V^* \mid h(r) = 0 \text{ for all } r \in R\}$.

Finally, note that $(A^!)^! \cong A$.

Example, the quadratic dual of the polynomial ring on two variables

Let x, y be a basis for V, with dual basis x^*, y^* for V^*.
Recall that R is the linear span of $x \otimes y - y \otimes x$.
Claim: $R^\perp = \mathrm{span}_k\{f \otimes f \mid f \in V^*\}$.
($\supseteq$) Let $f = ax^* + by^*$, $a, b \in k$. Then

$$\begin{aligned} f \otimes f &= (ax^* + by^*) \otimes (ax^* + by^*) \\ &= a^2 x^* \otimes x^* + ab(x^* \otimes y^* + y^* \otimes x^*) + b^2 y^* \otimes y^*. \end{aligned}$$

The last expression vanishes on $x \otimes y - y \otimes x$.
($\subseteq$) Let $g \in R^\perp$. Write

$$g = ax^* \otimes x^* + bx^* \otimes y^* + cy^* \otimes x^* + dy^* \otimes y^*.$$

Evaluating g on $x \otimes y - y \otimes x$ yields $b = c$. Now

$$\begin{aligned} g &= (a-b)x^* \otimes x^* + b(x^* + y^*) \otimes (x^* + y^*) + (d-b)y^* \otimes y^* \\ &\in \mathrm{span}_k\{f \otimes f \mid f \in V^*\}. \end{aligned}$$

Koszul Algebras (history)

Stewart Priddy introduced the notion of *Koszul algebra* in 1970. He was interested in computing the cohomology of topological spaces, specifically, he was studying the (stable) homotopy groups of spheres.
Priddy called them Koszul algebras in honor of Jean-Louis Koszul who had studied related notions for enveloping algebras of Lie algebras in the 1950s.

Koszul algebras (definition)

Recall the notion of graded projective resolution of ${}_A k$ for any k-algebra A.

$$\cdots \xrightarrow{f_2} P_2 \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \xrightarrow{\epsilon} k \longrightarrow 0$$

The P_i are graded free A-modules, so choosing bases for the P_i allows us to write the f_i as matrices with entries in A.

Definition

A graded algebra A is *Koszul* if ${}_A k$ has a graded projective resolution in which the matrix entries of all the f_i are linear (elements of A_1).

Koszul algebras and quadratic algebras

This definition is essentially Priddy's original definition; many equivalent conditions are known.

"Koszul" $\implies$ "quadratic", but the converse is false (as we will see soon).

Koszul algebras arise in many areas: topology, algebraic geometry (commutative and non-commutative), number theory, combinatorics.

Theorem (Priddy)

Let A be a quadratic algebra. Then A is Koszul if and only if $Ext_A(k,k) \cong A^!$.

Corollary

Let A be a Koszul algebra. Then $H_A(t)H_{A^!}(-t) = 1$.

So Koszul algebras are algebras in which: "Ext is easy to solve for" and we get information on Hilbert series.

Example, the polynomial ring on two variables

Let $A = k[x, y]$.

To prove A is Koszul we have to resolve ${}_A k$ and see what the maps look like. Using the fact that we have a nice basis for A, namely $\{x^i y^j \mid i, j \geq 0\}$, it is not hard to show

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} -y & x \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} A \xrightarrow{\epsilon} k \longrightarrow 0$$

is a projective resolution of ${}_A k$.

Therefore $k[x, y]$ is Koszul and $\mathrm{Ext}_A(k, k) \cong \Lambda(x^*, y^*)$ where x^*, y^* are dual to x, y respectively.

Computing an Ext product

Let's show that $x^* \star x^* = 0$.

$$
\begin{tikzcd}
A \arrow[r, "\begin{pmatrix} -y & x \end{pmatrix}"] \arrow[d, dashed, "\begin{pmatrix} 0 & -1 \end{pmatrix}"'] & A \oplus A \arrow[d, dashed, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] \arrow[dr, "x^*"] & \\
A \oplus A \arrow[r, "\begin{pmatrix} x \\ y \end{pmatrix}"'] \arrow[d, "x^*"'] & A \arrow[r, "\epsilon"'] & k \\
k & &
\end{tikzcd}
$$

Since the composition $\begin{pmatrix} 0 & -1 \end{pmatrix} x^* = 0$, we get $x^* \star x^* = 0$.

The algebra $\mathcal{U}(\mathfrak{g})$

Fix an integer $n \geq 2$.
Let V be the vector space over $\mathbb{Q}$ on the basis $\{X_{ij} \mid 1 \leq i \neq j \leq n\}$.

$$\begin{pmatrix} * & X_{12} & X_{13} & \cdots & X_{1n} \\ X_{21} & * & X_{23} & \cdots & X_{2n} \\ \vdots & & & & \vdots \\ X_{n1} & \cdots & \cdots & X_{nn-1} & * \end{pmatrix}$$

Let R be the subspace of $V \otimes V$ spanned by

$$\begin{aligned} &[X_{ij}, X_{kj}], \\ &[X_{ij}, X_{ik} + X_{jk}], \\ &[X_{ij}, X_{kl}], \end{aligned}$$

for all i, j, k, l distinct. (Here $[a, b] = ab - ba$.)
Define $\mathcal{U}(\mathfrak{g}) := T(V)/\langle R \rangle$ (enveloping algebra of a Lie algebra $\mathfrak{g}$).

Where does $\mathcal{U}(\mathfrak{g})$ come from?

It follows from work of Jensen-McCammond-Meier in 2006 that

$$\mathcal{U}(\mathfrak{g})^! = H^*(G, \mathbb{Q}) = \mathrm{Ext}_{\mathbb{Q}G}(\mathbb{Q}, \mathbb{Q}),$$

where G denotes the McCool group.

The McCool group is a group whose elements can be realized as either:

1) movies of unlinked circles moving in $\mathbb{R}^3$, or
2) certain automorphisms of a free group.

Question

Is $\mathcal{U}(\mathfrak{g})$ a Koszul algebra? (Raised by Cohen-Pruidze, Denham.)

Koszulity of $\mathcal{U}(\mathfrak{g})$

For $n = 2, 3$ it's not hard to prove $\mathcal{U}(\mathfrak{g})$ is Koszul.

Theorem (Conner-G)

For $n \geq 4$, $\mathcal{U}(\mathfrak{g})$ is not Koszul.

Some words about the proof.

1) It suffices to prove the result for $n = 4$ since there is a split injection $i : \mathcal{U}(\mathfrak{g})_n \to \mathcal{U}(\mathfrak{g})_{n+1}$.
2) There is a free Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. The Hochschild-Serre spectral sequence $\implies$ it suffices to show $\mathcal{U}(\mathfrak{g}/\mathfrak{h})$ is not Koszul.

Koszulity of $\mathcal{U}(\mathfrak{g})$

3) The Hilbert series of $\mathcal{U}(\mathfrak{g}/\mathfrak{h})^!$ is $(1+4t)^2$. So "$\mathcal{U}(\mathfrak{g}/\mathfrak{h})$ Koszul" $\implies H_{\mathcal{U}(\mathfrak{g}/\mathfrak{h})} = \frac{1}{(1-4t)^2}$.
4) We found that $H_{\mathcal{U}(\mathfrak{g}/\mathfrak{h})}$ agrees with $\frac{1}{(1-4t)^2} =$

   $$1+8t+48t^2+256t^3+1280t^4+6144t^5+28672t^6+131072t^7+\cdots,$$

   but, in degree 8, the coefficient of t^8 in $\frac{1}{(1-4t)^2}$ is 589824 whereas the coefficient of t^8 in $H_{\mathcal{U}(\mathfrak{g}/\mathfrak{h})}$ is 5898**34**.
5) The third matrix, M_3, in a minimal resolution of k contains elements of degree 6.

The fact that $\mathrm{Ext}^{3,8}_{\mathcal{U}(\mathfrak{g})}(k,k)$ is nonzero, implies that $\mathfrak{g}$ has some nontrivial crossed modules. However what these are remains mysterious, at the moment.

The algebras, $\mathcal{E}_n$

Fix an integer $n \geq 2$.

Let V be the k-vector space with basis $\{X_{ij} \mid 1 \leq i < j \leq n\}$.

Let R be the subspace of $V \otimes V$ spanned by

$$\begin{aligned}
&X_{ij}^2, \\
&X_{ij}X_{jk} - X_{jk}X_{ik} - X_{ik}X_{ij}, \\
&X_{jk}X_{ij} - X_{ik}X_{jk} - X_{ij}X_{ik}, \\
&[X_{ij}, X_{kl}],
\end{aligned}$$

for all $i < j < k$.

Define $\mathcal{E}_n := T(V)/\langle R \rangle$.

Divided difference operators

Where do the relations of $\mathcal{E}_n$ come from?

Let ∂_{ij} be the divided difference operator.

It acts on the polynomial ring $k[y_1, \ldots, y_n]$ by

$$\partial_{ij} f = \frac{f - s_{ij} f}{y_i - y_j}$$

where $s_{ij} f$ denotes the result of interchanging y_i and y_j in $f \in k[y_1, \ldots, y_n]$.

Now one checks that the operators ∂_{ij} satisfy the defining relations of $\mathcal{E}_n$.

One way of constructing the famous *Schubert polynomials* is by applying compositions of the ∂_{ij} to certain monomials.

Why are the $\mathcal{E}_n$ algebras interesting?

1) They were introduced by Fomin and Kirillov in connection with the cohomology of the flag manifold and the corresponding Schubert calculus.
2) There is a natural Hopf algebra structure on the twisted group algebra $\mathcal{E}_n \rtimes S_n$ where S_n denotes the symmetric group.
3) They fit into a larger framework of braided Hopf algebras associated to Coxeter groups (Milinski-Schneider), and they are related to Nichols algebras.

Open Question

Is $\mathcal{E}_n$ finite-dimensional for $n \geq 6$?

$$\dim_k(\mathcal{E}_2) = 2, \quad \dim_k(\mathcal{E}_3) = 12,$$
$$\dim_k(\mathcal{E}_4) = 576, \quad \dim_k(\mathcal{E}_5) = 8294400$$

Koszulity and $\mathcal{E}_n$

Theorem (Roos)

For $n \geq 3$, $\mathcal{E}_n$ is not a Koszul algebra.

However, these algebras do appear to be close to Koszul, in some sense.

Theorem (Conner-G)

Let R denote the algebra $\mathcal{E}_3$. Then

$$Ext_R(k,k) \cong R^![s], \quad Ext_{R^!}(k,k) \cong R[t]$$

for some classes $s \in Ext_R^{4,6}(k,k)$, $t \in Ext_{R^!}^{4,6}(k,k)$.

It seems that not much research has been done on the Ext algebras of the $\mathcal{E}_n$. Maybe this sheds some light on the finite-dimensionality question?

Thank you for listening!