Dividing a Square into Triangles: Combinatorial Topology meets the 2-adic Absolute Value

Peter Goetz

Mathematics Colloquium
CSU Chico

October 20, 2017
In 1967, in the American Mathematical Monthly:

5479. Proposed by Fred Richman and John Thomas, New Mexico State University

Let $N$ be an odd integer. Can a rectangle be dissected into $N$ nonoverlapping triangles, all having the same area?

By scaling, one may assume the rectangle is the unit square, $S$.

Assume $S$ has vertices:

$$(0,0),\quad (1,0),\quad (0,1),\quad (1,1).$$
A dissection of $S$ (or more generally any polygon) is a partition of $S$ into finitely many non-overlapping triangles.

A triangulation of $S$ is a dissection for which all of the triangles “fit together edge-by-edge”.

An $m$-equidissection is a dissection containing $m$ triangles all having the same area.

Evidently $m$-equidissections exist for any positive even integer $m$. 

...
**Theorems of Thomas, Monsky**

<table>
<thead>
<tr>
<th>Theorem (J. Thomas, 1968)</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>If there is an m-equidissection of the unit square such that the coordinates of all of the vertices of the triangles are rational numbers with odd denominators, then m is even.</em></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (P. Monsky, 1970)</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>If the unit square has an m-equidissection, then m is even.</em></td>
</tr>
</tbody>
</table>

For competitive audience members:

show there are no 3-equidissections of the unit square.
Goal: outline Monsky’s proof

Subgoal: introduce the $p$-adic numbers

Outline

I. $p$-adic absolute value and $p$-adic numbers
II. Sperner’s Lemma
III. Monsky’s proof
IV. Generalization and a question
V. References
A function $|\cdot| : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is an **absolute value** if:

(i) $|x| = 0$ iff $x = 0$,
(ii) $|xy| = |x| |y|$,
(iii) $|x + y| \leq |x| + |y|$.

Notice that if $|\cdot| : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ satisfies

(iii') $|x + y| \leq \max\{|x|, |y|\}$,

then $|\cdot|$ satisfies (iii).

Condition (iii') is called the **ultrametric inequality**.

Absolute values that satisfy the ultrametric inequality are called **non-Archimedean**. For a non-Archimedean absolute value:

$$|n| \leq 1 \text{ for all } n \in \mathbb{Z}.$$
"There are some, king Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand not only that which exists about Syracuse and the rest of Sicily

the sand so taken. But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way
Let \( | |\) be a non-Archimedean absolute value. If \( |x| \neq |y|\), then \( |x + y| = \max\{|x|, |y|\}\).

So in the non-Archimedean world all “triangles” are isoceles.

We will refer to this property as the **ATI-property**, abbreviated (ATI).

**Proof.**

First, \( |x + y| \leq \max\{|x|, |y|\} \). Assume that \( |x| < |y|\). Then \( \max\{|x|, |y|\} = |y| \). Now

\[
\max\{|x|, |y|\} = |y| = |x + y - x| \leq \max\{|x + y|, |x|\} = |x + y|
\]

(the last equality holds since \( |x| < |y|\)). We conclude that \( |x + y| = \max\{|x|, |y|\} \).
The $p$-adic absolute value

Fix $p$, a prime number.

For a nonzero rational number $x$, write $x = p^n \frac{a}{b}$ for some integers $n, a, b$ such that $p \nmid ab$. Then define:

$$|x|_p = \frac{1}{p^n}.$$  

Also set $|0|_p = 0$.

**Example:** $p = 2$:

$$|2|_2 = |2^1|_2 = \frac{1}{2}, \quad \left| \frac{1}{20} \right|_2 = \left| 2^{-2} \frac{1}{5} \right|_2 = 4$$

$$|2n + 1|_2 = 1, \quad \left| \frac{1}{2n + 1} \right|_2 = 1$$
The $p$-adic absolute value

Theorem

The function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is a non-Archimedean absolute value.
One can complete \( \mathbb{Q} \) with respect to \( | \cdot |_p \) to obtain

\[ \mathbb{Q}_p: \text{field of } p\text{-adic numbers.} \]

Elements of \( \mathbb{Q}_p \) are Laurent series in \( p \)

\[ a_{-n} p^{-n} + \cdots + a_0 p^0 + a_1 p^1 + a_2 p^2 + \cdots , \]

where the coefficients \( a_i \in \{0, 1, \cdots, p - 1\} \).

Contrast this with, for example:

\[ 10\pi = 3 \cdot 10^1 + 1 \cdot 10^0 + 4 \cdot 10^{-1} + 1 \cdot 10^{-2} + 5 \cdot 10^{-3} + \cdots . \]

Adding, subtracting, multiplying, and dividing in \( \mathbb{Q}_p \) is done by using the usual operations on series, and “carrying”.
The p-adic Numbers

The Freshman’s Dream

$$(a + b)^p = a^p + b^p,$$

for $a, b \in \mathbb{F}_p$.

The MATH 121 Student’s Dream

$$\sum_{n=1}^{\infty} x_n \text{ converges if and only if } \lim_{n \to \infty} x_n = 0,$$

where $x_n \in \mathbb{Q}_p$.

In $\mathbb{Q}_2$,

$$-1 = 1 + 2 + 2^2 + 2^3 + \cdots.$$

Recall:

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots.$$
Chevalley’s Extension Theorem

For the application to dissections:
extend the domain of $|\cdot|_2$ from $\mathbb{Q}$ to the field of real numbers, $\mathbb{R}$.

For example:

$$|\sqrt{2}|_2^2 = |(\sqrt{2})^2|_2 = |2|_2 = \frac{1}{2} \implies |\sqrt{2}|_2 = \frac{1}{\sqrt{2}},$$

but it’s not clear how we should define $|e|_2$ or $|\pi|_2$.

**Theorem (C. Chevalley)**

*Let $K \subseteq L$ be an extension of fields. Any non-Archimedean absolute value defined on $K$ can be extended to $L$.***

As a consequence: **we can extend** $|\cdot|_2$ **to** $\mathbb{R}$.

**From now on:*** $|\cdot|$ will denote an extension of $|\cdot|_2$ to $\mathbb{R}$. 
Sperner’s Lemma

- triangulated polygon
- each vertex is colored red, green, or blue
- **chromatic triangle**: triangle whose vertices are colored red, green and blue
- **red-blue edge**: an edge whose endpoints are colored red and blue

**Lemma (E. Sperner, 1928)**

*If the number of red-blue edges on the boundary is odd, then there is at least one chromatic triangle.*
Three Coloring the Plane

For any \((x, y) \in \mathbb{R}^2\),

\[(x, y)\] is colored

\[
\begin{cases}
  \text{red} & \text{if } |x| < 1, |y| < 1, \\
  \text{green} & \text{if } |x| < |y|, |y| \geq 1, \\
  \text{blue} & \text{if } |x| \geq |y|, |x| \geq 1.
\end{cases}
\]
Three Coloring the Plane

\[(x, y)\] is colored \[
\begin{array}{ll}
\text{red} & \text{if } |x| < 1, |y| < 1, \\
\text{green} & \text{if } |x| < |y|, |y| \geq 1, \\
\text{blue} & \text{if } |x| \geq |y|, |x| \geq 1.
\end{array}
\]

Colored unit square; points have coordinates \(k/20\)
Translation invariance

**Lemma**

Translating any point $P$ by $A$ does not change the color of $P$.

- $A = (a_1, a_2)$: $|a_1| < 1$ and $|a_2| < 1$

- assume $P = (x, y)$ is blue: so $|x| \geq |y|$ and $|x| \geq 1$

- to prove: $P + A = (x + a_1, y + a_2)$ is blue,
  
  $|x + a_1| \geq |y + a_2|$ and $|x + a_1| \geq 1$

- first, $|x| > |a_1|$, so (ATI) $\implies |x + a_1| = |x| \geq 1$

- second, $|y + a_2| \leq \max\{|y|, |a_2|\} \leq |x| = |x + a_1|$

- therefore $P + A$ is blue
Lines are not Chromatic

Lemma

Any line in the plane cannot contain points of all three colors.
Lemma

Any line in the plane cannot contain points of all three colors.

- Suppose some line contains a red point, a blue point and a green point.
- By translating the red point to the origin we can assume the line passes through the origin.
- Now let \((x, y)\) denote a blue point; let \((x', y')\) denote a green point.
- Then \(|y| \leq |x|\) and \(|x'| < |y'|\), so \(|x'y| < |xy'|\).
- The points are on the same line, so \(x'y = xy'\), so \(|x'y| = |xy'|\), which is a contradiction.
Chromatic Triangles Exist

**Theorem**

*Every dissection of the colored unit square contains an odd number of chromatic triangles.*
Chromatic Triangles Exist

- **the key fact:** lines contain points of at most two colors
- consider red-blue edges
- there is an *odd* number of red-blue edges between a red vertex and a blue vertex
- there is an *even* number of red-blue edges between any two vertices with a color combination different than red, blue

**therefore:**
- bottom boundary contains an *odd* number of red-blue edges
- other boundary edges contain *no* red-blue edges
Chromatic Triangles Exist

- chromatic triangles: *odd* number of red-blue edges in their boundaries
- non-chromatic triangles: *even* number of red-blue edges in their boundaries
- **count**: red-blue edges in boundaries of all triangles:
  - red-blue edges in the interior get counted twice
  - there is an *odd* number of red-blue edges on the boundary
  - thus the **count** is *odd*
- therefore, there must be an *odd* number of chromatic triangles
Lemma

Let $T$ be a chromatic triangle. Then $|\text{area}(T)| \geq 2$.

By translation invariance, we may assume that $T$’s red vertex is $(0, 0)$. Let $B = (b_1, b_2)$ and $C = (c_1, c_2)$ be the other two vertices.

\[
\text{area}(T) = \pm \frac{1}{2} \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} = \pm \frac{1}{2} (b_1c_2 - b_2c_1)
\]

Now

\[
|b_1| \geq |b_2| \text{ and } |b_1| \geq 1;
\]

\[
|c_1| < |c_2| \text{ and } |c_2| \geq 1,
\]

so $|b_1c_2| > |b_2c_1|$. Then, using the (ATI) property,

\[
|\text{area}(T)| = \left| \frac{1}{2} \right| |b_1c_2 - b_2c_1| = 2 |b_1| |c_2| \geq 2.
\]
Theorem (P. Monsky, 1970)

If the unit square has an $m$-equidissection, then $m$ is even.
Theorem (P. Monsky, 1970)

If the unit square has an $m$-equidissection, then $m$ is even.

Proof.

Assume that we have an $m$-equidissection of the unit square.
Monsky’s Theorem

Theorem (P. Monsky, 1970)

*If the unit square has an \( m \)-equidissection, then \( m \) is even.*

Proof.

Assume that we have an \( m \)-equidissection of the unit square. Let \( T \) be a chromatic triangle in the dissection.
Theorem (P. Monsky, 1970)

If the unit square has an $m$-equidissection, then $m$ is even.

Proof.

Assume that we have an $m$-equidissection of the unit square. Let $T$ be a chromatic triangle in the dissection. The unit square has area equal to 1, so $\text{area}(T) = 1/m$, therefore $|1/m| \geq 2$. 
Monsky’s Theorem

Theorem (P. Monsky, 1970)

If the unit square has an $m$-equidissection, then $m$ is even.

Proof.

Assume that we have an $m$-equidissection of the unit square. Let $T$ be a chromatic triangle in the dissection.
The unit square has area equal to 1, so $\text{area}(T) = 1/m$, therefore $|1/m| \geq 2$.
Recall that for any odd integer $n$, $|1/n| = 1$. 

Theorem (P. Monsky, 1970)

*If the unit square has an* \( m \)-*equidissection*, then \( m \) is even.

**Proof.**

Assume that we have an \( m \)-equidissection of the unit square. Let \( T \) be a chromatic triangle in the dissection. The unit square has area equal to 1, so \( \text{area}(T) = 1/m \), therefore \( |1/m| \geq 2 \).

Recall that for any odd integer \( n \), \( |1/n| = 1 \).

It follows that \( m \) is even.
Sherman Stein conjectured around 1989:
Any centrally symmetric subset $S$ of $\mathbb{R}^2$ has no odd-equidissections.

**Theorem (P. Monsky, 1990)**

*Let $S$ be a centrally symmetric subset of $\mathbb{R}^2$. Then every equidissection of $S$ is even.*

As a consequence: all equidissections of regular $2n$-gons are even.
Sherman Stein’s Question (2004):

Does a trapezoid whose parallel edges are in ratio $\sqrt{2}$ to 1 have any equidissections?
References


References


https://math.berkeley.edu/~moorxu/misc/equiareal.pdf
Thank you for listening!
Theorem (E. Kasimatis, 1989)

*If there is an $m$-equidissection of a regular $n$-gon, $n \geq 5$, then $m$ is a multiple of $n$.***
Theorem (D. Mead, 1979)

Let \( n \geq 3 \). Suppose the unit \( n \)-dimensional cube is dissected into \( m \) simplices all having the same volume. Then \( m \) is a multiple of \( n! \).
If one makes a similar definition as in the $p$-adic absolute value for $|\cdot|_6$, then the resulting function is **not** multiplicative:

$$|2|_6 = 1, |3|_6 = 1, \text{ and } |6|_6 = \frac{1}{6} \neq |2|_6 |3|_6 = 1.$$
A remark on the theorem of Chevalley

The hypothesis that $\lvert \lvert$ is non-Archimedean in the statement of Chevalley’s Theorem is necessary. Consider the extension $\mathbb{C} \subseteq \mathbb{C}(t)$.

**Theorem (Ostrowski)**

*Let $K$ be a field which is complete with respect to an Archimedean absolute value $\lvert \lvert$. Then $K$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ and $\lvert \lvert$ is equivalent to the ordinary absolute value.*

Suppose that the usual (Archimedean) absolute value on $\mathbb{C}$ extends to $\mathbb{C}(t)$. Then $\mathbb{C}(t)$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

First of all, $\mathbb{C}$ is a subfield of $\mathbb{C}(t)$ and there is no square root of $-1$ in $\mathbb{R}$, so $\mathbb{C}(t)$ cannot be isomorphic to $\mathbb{R}$.

The other alternative leads to a contradiction: $\mathbb{C}$ is algebraically closed; $t$ has no square root in $\mathbb{C}(t)$, so $\mathbb{C}(t)$ is not algebraically closed, therefore $\mathbb{C}(t)$ is not isomorphic to $\mathbb{C}$. 