Bézout's Theorem: A Tale of Two Curves

Peter Goetz

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Saint Mary's College of California

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Outline

1) Introduction
2) Complex Solutions
3) The Projective Plane
4) Intersection Number
5) Bézout's Theorem

$f(x, y)$, a polynomial over the complex numbers, $\mathbb{C}$

Ex. $f(x, y) = 1 + x + iy^2 - x^2 - x^3y$, $\deg(f(x, y)) = 4$

$f(x, y)$ determines a complex-valued function on the complex plane, $\mathbb{C}^2$

An **affine algebraic curve**, C, is the zero set of some $f(x, y)$, i.e.,

$$C = \{(a, b) \in \mathbb{C}^2 \mid f(a, b) = 0\}.$$

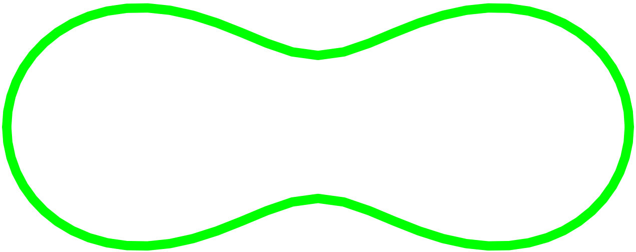
Define **degree of** C to be $\deg(f(x, y))$.

Folium of Descartes

Ex. $x^3 + y^3 - 6xy$, degree: 3

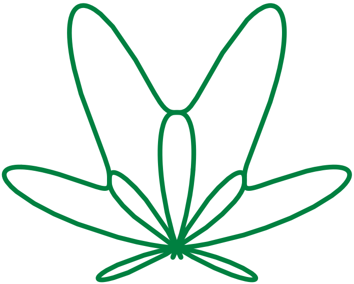
Oval of Cassini

Ex. : $(x^2 + y^2 + 9)^2 - 36x^2 - 100$, degree: 4



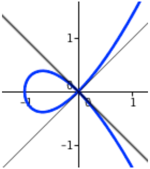
Frog Curve

Ex. $r - \sin\theta - \sin^3 4\theta$, degree: 26



Multiplicity of p

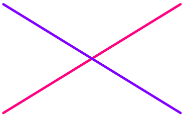
$p = (0, 0)$
Write $f = f_m + f_{m+1} + \cdots + f_k$, where f_i is **homogeneous** of degree i, $f_m \neq 0$.
The **multiplicity of** p **on** f, $m_p(f)$, is defined to be m.
The **tangent lines at** p are determined by the linear factors of f_m.
Ex. Nodal cubic: $y^2 - x^2 - x^3$



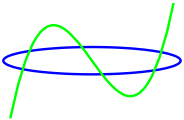
Two Curves

Problem: Given two algebraic curves C and D, count the number of points in $C \cap D$.

Ex. Two lines: $1 = 1 \cdot 1$



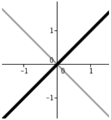
Ex. Ellipse and Cubic: $6 = 2 \cdot 3$



Elegant Guess: Suppose $\deg(C) = m$, $\deg(D) = n$. Then $|C \cap D| = mn$.

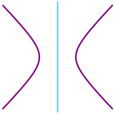
Examples

Ex. $y^2 - x^2$, $y - x$, $\infty \neq 2 \cdot 1$



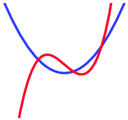
From now on, assume that C and D have no common components.

Ex. Line and Hyperbola: $0 \neq 1 \cdot 2$

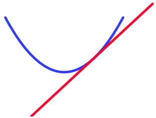


Examples

Ex. Parabola and Cubic: $3 \neq 2 \cdot 3$

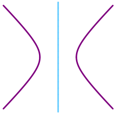


Ex. Parabola and Tangent Line: $1 \neq 2 \cdot 1$



Complex Solutions

Ex. $x^2 - y^2 = 1$, $x = 0$.

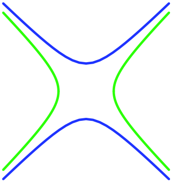


Substituting $x = 0$ yields $y^2 = -1$, so we get two *complex* points: $(0, \pm i)$.

Therefore we should count points in $\mathbb{C}^2$, and remember that we can draw pictures in $\mathbb{R}^2$.

Points at Infinity?

Ex. Two Hyperbolas: $x^2 - y^2 = 1$, $y^2 - x^2 = 1$



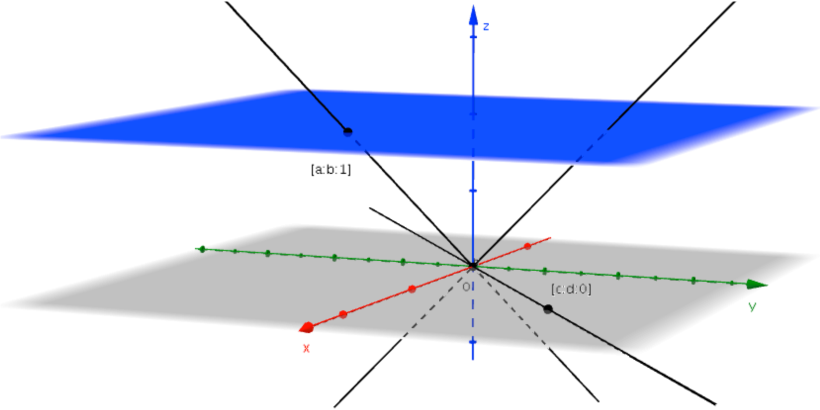
It is easy to check that there are no complex points on both hyperbolas.

Where might we find four points?

Idea: Enlarge the plane and add "points at ∞" corresponding to directions.

The Complex Projective Plane

Let $\mathbb{P}^2$ denote the set of all lines through the origin in $\mathbb{C}^3$.



Homogeneous Coordinates

A point $(a, b, c) \neq (0, 0, 0)$ in $\mathbb{C}^3$ determines a unique line through the origin.

(a, b, c), (a', b', c'), determine the same line if and only if there is some $0 \neq \lambda \in \mathbb{C}$ such that $(a, b, c) = \lambda(a', b', c')$.

Write $[a : b : c]$ for the equivalence class of (a, b, c).

$$\begin{aligned} \mathbb{P}^2 &= \left\{[a : b : c] \mid (0,0,0) \neq (a,b,c) \in \mathbb{C}^3\right\} \\ &= \mathbb{C}^2 \cup \mathbb{P}^1, \text{ where} \end{aligned}$$

$$\mathbb{C}^2 \leftrightarrow \{[a : b : 1] \mid a, b \in \mathbb{C}\}$$

$$\mathbb{P}^1 \leftrightarrow \{[a : b : 0] \mid a, b \in \mathbb{C}, \text{ not both zero}\}$$

Homogenous Polynomials on $\mathbb{P}^2$

In general, a polynomial $f(x, y, z)$ will not define a function on $\mathbb{P}^2$:

$$f(x, y, z) = x^2 - y + z; f([1 : 1 : 1]) = 1, f([2 : 2 : 2]) = 4.$$

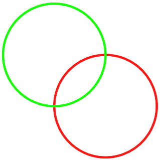
However, if $F(x, y, z)$ is **homogenous** of degree d (all terms in F have degree d) and $F([a : b : c]) = 0$, then

$$F([\lambda a : \lambda b : \lambda c]) = \lambda^d F([a : b : c]) = 0.$$

A **projective plane curve** is the zero set of a *homogeneous* polynomial $F(x, y, z)$.

Two Circles

Ex. $C : (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$; $D : x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$.



Visually, there are two points of intersection in $\mathbb{C}^2$.

Homogenize:

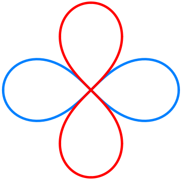
$C^* : (x - \frac{1}{2}z)^2 + y^2 = \frac{1}{4}z^2$
$D^* : x^2 + (y - \frac{1}{2}z)^2 = \frac{1}{4}z^2$

$z = 0 \Rightarrow x^2 + y^2 = 0$, yielding two intersection points $[1 : \pm i : 0]$

$4 = 2 \cdot 2$

Lemniscates of Bernoulli

Ex. $C : (x^2 + y^2)^2 + y^2 - x^2 = 0$; $D : (x^2 + y^2)^2 + x^2 - y^2 = 0$



The only complex point of intersection is $(0, 0)$.

There are only two points at ∞, namely $[1 : \pm i : 0]$.

Now $4 \cdot 4 = 16$, so we would like to count sixteen points of intersection.

Somehow, we have to count the **multiplicities** of the intersection points.

Axioms for Intersection Number

Let $p \in \mathbb{C}^2$, and let C and D be affine plane curves.

Axioms:

1. $I(p, C \cap D)$ is a nonnegative integer or ∞.
2. $I(p, C \cap D) = 0$ if and only if $p \notin C \cap D$.
3. If $p = T(q)$ for some affine transformation T, then $I(p, C \cap D) = I(q, C^T \cap D^T)$.
4. $I(p, C \cap D) = I(p, D \cap C)$.
5. $I(p, C \cap D) \geq m_p(C)m_p(D)$, with equality if and only if C and D have no common tangent lines at p.
6. If $C = C_1C_2$, then $I(p, C \cap D) = I(p, C_1 \cap D) + I(p, C_2 \cap D)$.
7. $I(p, C \cap D) = I(p, C \cap (D + AC))$ for any polynomial $A(x, y)$.

Existence and Uniqueness of $I(p, C \cap D)$

It is not too hard to show that if $I(p, C \cap D)$ exists, then it is unique.

In fact, assuming $I(p, C \cap D)$ exists, there is an inductive proof showing that there is a finite algorithm for computing $I(p, C \cap D)$.

The hard part is to prove that $I(p, C \cap D)$ exists.

Next, we are going to show that algebra gives a way to rigorously define $I(p, C \cap D)$. Then it can be proved that our definition satisfies the seven axioms.

Algebra-Geometry Correspondence

Recall the set of polynomials, $R = \mathbb{C}[x, y]$, is a commutative ring (also a vector space over $\mathbb{C}$).

For a subset $I \subseteq \mathbb{C}[x, y]$, let

$$V(I) = \{(a, b) \in \mathbb{C}^2 \mid f(a, b) = 0 \text{ for every } f(x, y) \in I\}.$$

For a subset $X \subseteq \mathbb{C}^2$, let

$$I(X) = \{f(x, y) \in R \mid f(a, b) = 0 \text{ for every } (a, b) \in X\}.$$

Hilbert's Nullstellensatz:

$$\Big\{\text{algebraic sets, } V(I)\Big\} \rightleftarrows \Big\{\text{radical ideals, } \sqrt{I}\Big\}.$$

The collection of all algebraic sets form the closed sets of a topology on $\mathbb{C}^2$ called the **Zariski topology**.

Rings of Functions on Algebraic Sets

Let $X = V(I)$ be an algebraic set. The **coordinate ring of** X is the quotient ring $\mathbb{C}[x, y]/I(X)$.

We should think of $\mathbb{C}[x, y]/I(X)$ as the ring of polynomial functions defined on X.

Ex. $C = V(y - x^2)$, coordinate ring: $\mathbb{C}[x, y]/(y - x^2)$. Note that y and x^2 are the same function if we restrict their domains to C.

Local Properties

We want to study local properties: $m_p(C)$ or $I(p, C \cap D)$, of a curve or curves at a point p.

The right things to consider are the rational functions on the curve which are defined at p.

We get **local rings:** $\mathcal{O}_p(C)$. Local rings have a unique maximal ideal; in this case, that maximal ideal is denoted by $\mathfrak{m}_p(C)$.

Theorem: Let p be a point on an irreducible curve C. Then for all sufficiently large n,

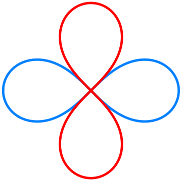
$$m_p(C) = \dim(\mathfrak{m}_p(C)^n / \mathfrak{m}_p(C)^{n+1}).$$

Define $I(p, C \cap D) = \dim(\mathcal{O}_p(\mathbb{C}^2)/(C, D))$.

Theorem: With this definition, $I(p, C \cap D)$ satisfies axioms (1)-(7) for an intersection number.

Lemniscates of Bernoulli

Ex. $C : (x^2 + y^2)^2 + y^2 - x^2$; $D : (x^2 + y^2)^2 + x^2 - y^2$



Points of Intersection in $\mathbb{P}^2$:

$p = (0, 0) \leftrightarrow [0 : 0 : 1]$, $q_1 = [1 : i : 0]$, $q_2 = [1 : -i : 0]$

$$I(p, C \cap D) + I(q_1, C \cap D) + I(q_2, C \cap D) = 16?$$

$I(p, C \cap D)$

$C : (x^2 + y^2)^2 + y^2 - x^2$; $D : (x^2 + y^2)^2 + x^2 - y^2$

$C + D = 2(x^2 + y^2)^2$, so $m_p(C + D) = 4$,
tangent lines at p: $y = \pm ix$.

$m_p(C) = 2$, tangent lines at p: $y = \pm x$

Then

$$\begin{aligned} I(p, C \cap D) &= I(C \cap (C + D)) && \text{by (7)} \\ &= m_p(C) m_p(C + D) && \text{by (5)} \\ &= 4 \cdot 2 = 8. \end{aligned}$$

$I(q_{\{1,2\}}, C \cap D)$

$C^* : (x^2+y^2)^2 + (y^2-x^2)z^2$; $D^* : (x^2+y^2)^2 + (x^2-y^2)z^2$
Set $x = 1$:
$C^*(1,y,z) = (1+y^2)^2 + (y^2-1)z^2$,
$D^*(1,y,z) = (1+y^2)^2 + (1-y^2)z^2$
Translate $(i,0)$ to the origin by the substitutions: $y \mapsto y - i$, $z \mapsto z$.
$C' : y^4 + y^2z^2 - 4iy^3 - 2iyz^2 - 2(2y^2+z^2)$,
$D' : y^4 - y^2z^2 - 4iy^3 + 2iyz^2 - 2(2y^2-z^2)$
Then

$$\begin{aligned} I(q_1, C \cap D) &= I((0,0), C' \cap D') && \text{by (3)} \\ &= m_{(0,0)}(C')m_{(0,0)}(D') && \text{by (5)} \\ &= 2 \cdot 2 = 4. \end{aligned}$$

By symmetry, $I(q_2, C \cap D) = 4$.
Therefore,

$$I(p, C \cap D) + I(q_1, C \cap D) + I(q_2, C \cap D) = 8 + 4 + 4 = 16.$$

Bézout's Theorem

Bézout's Theorem: Let C and D be projective plane curves of degrees m and n respectively. Suppose that C and D have no common components. Then

$$\sum_{p \in C \cap D} I(p, C \cap D) = mn.$$

History

Isaac Newton in his Principia, Book I, Lemma 28:

"*Hence it is that the intersections of the conic sections with the curves of the third order, because they may amount to six, ...; and the intersections of two curves of the third order, because they may amount to nine, ...*"

Étienne Bézout, in 1779, published the theorem in *Théorie générale des équations algébriques*.

Georges-Henri Halphen, in 1873, gave the, generally accepted, first correct proof.

Thank you for listening!

References

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2) R. Hartshorne, *Algebraic Geometry*

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