## Math 340, Number Theory, Fall 2016

## Homework 4 Solutions

**2.1.6** Compute $\tau(n)$ for $n$ between 21 and 30.

**Solution:**

| $n$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\tau(n)$ | 4 | 4 | 2 | 8 | 3 | 4 | 4 | 6 | 2 | 8 |

**2.2.2** Let $E$ denote the set of even integers. List the primes in $E$ between 21 and 40.

**Solution:** Note that $x \in E$ is prime in $E$ if and only if 4 does not divide $x$. So the primes in $E$ between 21 and 40 are 22, 26, 30, 34, and 38.

**2.2.18** Let $T$ be the set of positive integers of the form $3n + 1$. Give an example of distinct primes $p$, $q$, and $r$ in $T$ such that $p^2 = qr$.

**Solution:** Let $p = 10$, $q = 4$ and $r = 25$. Then $p$, $q$, and $r$ are primes in $T$ and $p^2 = qr$.

The next two problems refer to the set $S = \{a + b\sqrt{-6} \mid a, b \in \mathbb{Z}\}$.

**2.3.16** Show that if $A$ divides 1, then $A$ is 1 or $-1$.

*Proof.* Let $A \in S$. Suppose that $A$ divides 1 in $S$. Then $1 = AB$ for some $B \in S$. So $1 = |1|^2 = |AB|^2 = |A|^2|B|^2$. In particular (since $|A|^2$ is a nonnegative integer), $|A|^2 = 1$. Therefore $A$ is 1 or $-1$. □

**2.3.20** Show that $3+\sqrt{-6}$ is prime in $S$.

*Proof.* We will prove this by contradiction. Suppose that $3+\sqrt{-6}$ is not prime in $S$.

Then $3+\sqrt{-6} = AB$ for some $A$ and $B$ in $S$ with neither $A$ nor $B$ equal to 1 or $-1$. So $15 = |3+\sqrt{-6}|^2 = |AB|^2 = |A|^2|B|^2$. Hence $|A|^2$ is equal to 3 or 5. If we write $A = a + b\sqrt{-6}$ for some integers $a$ and $b$, then we would have $a^2+6b^2$ equal to 3 or 5. However it is easy to see by inspection that either of these two options is an impossibility. We are forced to conclude that $3+\sqrt{-6}$ is prime in $S$, as desired. $\square$