Math 340, Number Theory, Fall 2016
Homework 4 Solutions

2.1.6 Compute \( \tau(n) \) for \( n \) between 21 and 30.

Solution:

\[
\begin{array}{c|cccccccc}
 n & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\
\hline
\tau(n) & 4 & 4 & 2 & 8 & 3 & 4 & 4 & 6 & 2 & 8 \\
\end{array}
\]

2.2.2 Let \( E \) denote the set of even integers. List the primes in \( E \) between 21 and 40.

Solution: Note that \( x \in E \) is prime in \( E \) if and only if 4 does not divide \( x \). So the primes in \( E \) between 21 and 40 are 22, 26, 30, 34, and 38.

2.2.18 Let \( T \) be the set of positive integers of the form \( 3n + 1 \). Give an example of distinct primes \( p, q, \) and \( r \) in \( T \) such that \( p^2 = qr \).

Solution: Let \( p = 10, q = 4 \) and \( r = 25 \). Then \( p, q, \) and \( r \) are primes in \( T \) and \( p^2 = qr \).

The next two problems refer to the set \( S = \{ a + b\sqrt{-6} \mid a, b \in \mathbb{Z} \} \).

2.3.16 Show that if \( A \) divides 1, then \( A \) is 1 or \( -1 \).

Proof. Let \( A \in S \). Suppose that \( A \) divides 1 in \( S \). Then \( 1 = AB \) for some \( B \in S \). So \( 1 = |1|^2 = |AB|^2 = |A|^2|B|^2 \). In particular (since \( |A|^2 \) is a nonnegative integer), \( |A|^2 = 1 \). Therefore \( A \) is 1 or \( -1 \). \( \square \)
2.3.20 Show that $3 + \sqrt{-6}$ is prime in $S$.

Proof. We will prove this by contradiction. Suppose that $3 + \sqrt{-6}$ is not prime in $S$.

Then $3 + \sqrt{-6} = AB$ for some $A$ and $B$ in $S$ with neither $A$ nor $B$ equal to $1$ or $-1$. So $15 = |3 + \sqrt{-6}|^2 = |AB|^2 = |A|^2|B|^2$. Hence $|A|^2$ is equal to $3$ or $5$. If we write $A = a + b\sqrt{-6}$ for some integers $a$ and $b$, then we would have $a^2 + 6b^2$ equal to $3$ or $5$. However it is easy to see by inspection that either of these two options is an impossibility. We are forced to conclude that $3 + \sqrt{-6}$ is prime in $S$, as desired. □