Math 340, Number Theory, Fall 2016

Homework 3 Solutions

1.5.15 Find the least residue of b modulo m where $m = 7$, $b = -100$.

Solution: We have $-100 \equiv 5 \pmod 7$. The least residue of -100 modulo 7 is 5.

1.5.30 Suppose $a \equiv 2 \pmod{17}$, $b \equiv 4 \pmod{17}$, and $c \equiv 5 \pmod{17}$. What is the least residue of $a^2 + b^2 + c^2 \pmod{17}$?

Solution: We compute
$a^2 + b^2 + c^2 \equiv (2)^2 + (4)^2 + (5)^2 \equiv 4 + 16 + 25 \equiv 45 \equiv 11 \pmod{17}$. So the least residue of $a^2 + b^2 + c^2$ mod 17 is 11.

1.6.16 Compute $f_1 + f_2 + \cdots f_n$ for $n = 1, 2, 3, 4, 5, 6$. Conjecture a simpler formula for the sum.

Solution: Let $S(n) = f_1 + f_2 + \cdots f_n$. We have the following table:

n	1	2	3	4	5	6
$S(n)$	1	2	4	7	12	20

We note that the numbers in the bottom row are 1 less than Fibonacci numbers. We conjecture that $S(n) = f_{n+2} - 1$.

1.6.22 Prove that $16^n \equiv 1 - 10n \pmod{25}$ for all positive integers n.

Proof. For the base step we note $16^1 \equiv 1 - 10(1) \pmod{25}$. For the inductive step we assume $16^k \equiv 1 - 10k \pmod{25}$ for some positive integer k. We note $16 \equiv -9 \pmod{25}$ and $160 \equiv 10 \pmod{25}$. Now consider

$$\begin{aligned} 16^{k+1} &\equiv 16(1 - 10k) \pmod{25} \\ &\equiv 16 - 160k \pmod{25} \\ &\equiv -9 - 10k \pmod{25} \\ &\equiv 1 - 10(k+1) \pmod{25}. \end{aligned}$$

This verifies the inductive step. We conclude that $16^n \equiv 1 - 10n \pmod{25}$ for all positive integers n. □

1.6.32 Prove that $f_1 + f_2 + \cdots f_n = f_{n+2} - 1$ for all integers $n \geq 1$.

Proof. We use induction on n. In a previous problem we verified that $f_1 + f_2 + \cdots f_n = f_{n+2} - 1$ for all $n = 1, 2, 3, 4, 5, 6$. Now assume $f_1 + f_2 + \cdots f_k = f_{k+2} - 1$ for some $k \geq 6$. Then $f_1 + f_2 + \cdots f_k + f_{k+1} = f_{k+2} - 1 + f_{k+1} = f_{k+3} - 1 = f_{(k+1)+2} - 1$. By mathematical induction, $f_1 + f_2 + \cdots f_n = f_{n+2} - 1$ for all integers $n \geq 1$. □