Math 340, Number Theory, Fall 2016

Homework 2 Solutions

1.3.10 Find x and y such that $27x + 15y = (27, 15)$ with x positive but as small as possible.

Solution. Using the Euclidean algorithm we have
$(27, 15) = 3 = (-1)27 + (2)15$. So all solutions of $27x + 15y = 3$ are given by $x = -1 + (15/3)t$ and $y = 2 - (27/3)t$. The smallest positive x occurs when $t = 1$ and $x = 4$. So the solution is $x = 4$ and $y = -7$.

1.3.28 Let a, b be positive integers. Show that there exist integers x and y such that $\frac{1}{[a,b]} = \frac{x}{a} + \frac{y}{b}$.

Proof. There exist integers x and y such that $(a, b) = ya+xb$. Now consider

$$\begin{aligned} \frac{1}{[a,b]} &= \frac{(a,b)}{ab} \\ &= \frac{ya + xb}{ab} \\ &= \frac{x}{a} + \frac{y}{b}. \end{aligned}$$

This completes the proof. □

1.3.29 Prove that if $(a, b) = 1$ and $c \neq 0$, then $(ac, bc)|c$.

Proof. We assume that $(a, b) = 1$ and $c \neq 0$. Then there exist integers x and y such that $1 = ax + by$. Multiplying by c we get $c = acx + bcy$. Let $d = (ac, bc)$ (note that the hypothesis $c \neq 0$ is used here to ensure d exists). Then d divides ac and d divides bc. So certainly d divides $acx + bcy$, i.e., d divides c. This completes the proof. □

1.4.24 A roadside stand bought 11 large baskets of eggs from a farmer and sold 39 small baskets of eggs, which hold fewer than a dozen. There were 19 eggs left over. How many eggs does a large basket hold?

Solution. Let x be the number of eggs in a large basket. Let y be the number of eggs in a small basket. Then $11x = 39y + 19$. Rearranging we have

$$11x - 39y = 19 \quad (1).$$

Using the Euclidean algorithm we have $1 = (-7)(11) + (-2)(-39)$. Therefore a particular solution of (1) is $x_0 = -133$ and $y_0 = -38$. So all solutions of (1) are given by $x = -133 - 39t$, $y = -38 - 11t$ as t ranges over $\mathbb{Z}$. The inequality $-38 - 11t > 0$ leads to $t \leq 4$. The inequality $-38 - 11t < 12$ leads to $t \geq 4$. Hence $t = 4$ and $x = 23$. So there are 23 eggs in a large basket.

1.4.36 Prove that for integers a, b, c if $a|bc$, then $a|(a,b)c$ (provided (a,b) exists).

Proof. Let a, b, c be integers such that $a|bc$ and (a,b) exists. Let x and y be integers such that $(a,b) = xa + yb$. Multiplying through by c yields $(a,b)c = xac + ybc$. Certainly $a|xac$ and $a|ybc$ follows from our assumption. Hence $a|xac + ybc = (a,b)c$.

□