Math 240: Introduction to Mathematical Thought

Homework 8, Solutions

Assigned exercises

6.16 Prove that $7|3^{4n+1} - 5^{2n-1}$ for every positive integer n.

Proof. We use induction. For the base step, $3^{4(1)+1} - 5^{2(1)-1} = 238 = 7(34)$, so $7|3^{4(1)+1} - 5^{2(1)-1}$.

For the inductive step, assume that $7|3^{4k+1} - 5^{2k-1}$ for some positive integer k. Then $3^{4k+1} - 5^{2k-1} = 7m$ for some $m \in \mathbb{Z}$. Then

$$\begin{aligned}
3^{4(k+1)+1} - 5^{2(k+1)-1} &= 3^{4k+1+4} - 5^{2k-1+2} \\
&= 3^4(3^{4k+1} - 5^{2k-1}) + 3^4 \cdot 5^{2k-1} - 5^{2k-1+2} \\
&= 3^4(3^{4k+1} - 5^{2k-1}) + 5^{2k-1}(3^4 - 5^2) \\
&= 3^4(7m) + 5^{2k-1} \cdot 7 \cdot 8 \\
&= 7(3^4 m + 5^{2k-1} \cdot 8).
\end{aligned}$$

This verifies the inductive step.

We conclude by induction that $7|3^{4n+1} - 5^{2n-1}$ for every positive integer n. $\square$

6.24 Prove Bernoulli's Identity: For every real number $x > -1$ and every positive integer n,

$$(1+x)^n \geq 1 + nx.$$

Proof. Fix a real number $x > -1$. The base step, $n = 1$, is clear:

$$(1+x)^1 = 1 + x = 1 + (1)x.$$

For the inductive step, assume that k is a positive integer and $(1+x)^k \geq 1 + kx$. Since $x > -1$ we know that $1 + x > 0$. We multiply the assumed inequality on both sides by the *positive* quantity $1 + x$ to get:

$$(1+x)^k(1+x) \geq (1+kx)(1+x).$$

Therefore

$$(1+x)^{k+1} \geq 1 + x + kx + kx^2 \geq 1 + x + kx = 1 + (k+1)x,$$

where the second inequality holds since $kx^2 \geq 0$. This completes the inductive step.

We conclude by the Principle of Mathematical Induction that the desired inequality holds for every positive integer n. $\square$

6.42 A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 2$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Conjecture a formula for a_n and verify your conjecture.

We have $a_3 = 4$ and $a_4 = 8$, so we conjecture that $a_n = 2^{n-1}$ for all positive integers n.

Proof. We use strong induction. The first four cases have been confirmed above. Let $k \geq 2$ be an integer and assume that for all integers i, $1 \leq i \leq k$, that $a_i = 2^{i-1}$. Then

$$a_{k+1} = a_k + 2a_{k-1} = 2^{k-1} + 2 \cdot 2^{k-2} = 2^{k-2}(2+2) = 2^{k-2} \cdot 2^2 = 2^{(k+1)-1}.$$

This verifies the inductive step.

We conclude, by induction, that $a_n = 2^{n-1}$ for all positive integers n. □

Extra exercises

6.13 Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for every positive integer n.

Proof. The base step, $n = 1$, is clear:

$$1 \cdot 1! = 1, \text{ and } (1+1)! - 1 = 1.$$

For the inductive step, assume that k is a positive integer and

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1.$$

Now we have

$$\begin{aligned} 1 \cdot 1! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)!(1 + k + 1) - 1 \\ &= (k+2)! - 1 \\ &= ((k+1) + 1)! - 1. \end{aligned}$$

This completes the inductive step.

We conclude the result by the Principle of Mathematical Induction. □

6.21 Prove that $4 \mid (5^n - 1)$ for every nonnegative integer n.

Proof. See class notes. □

6.45 Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 12$, there are nonnegative integers a and b such that $n = 3a + 7b$.

Proof. See class notes. □