Math 240: Introduction to Mathematical Thought

Homework 7, Solutions

Assigned exercises

5.40 Show that there exists a rational number a and an irrational number b such that a^b is rational.

Proof. Let $a = 1$ and $b = \sqrt{2}$. Note that $1^{\sqrt{2}} = 1$ is a rational number. □

5.50 Disprove the statement: There is a real number x such that $x^6 + x^4 + 1 = 2x^2$.

Disproof. Suppose there is a real number solution of the equation. Let $x \in \mathbb{R}$ be a solution. Then

$$x^6 + x^4 - 2x^2 + 1 = 0,$$

so

$$x^6 + (x^2 - 1)^2 = 0.$$

The left side of the last equation is a sum of two non-negative real numbers, so we must have $x^6 = 0$ and $(x^2 - 1)^2 = 0$. It then follows that $x = 0$ and $x = \pm 1$, which is a contradiction.

6.6 (a) What does $1^3 + 2^3 + \cdots + n^3$ represent geometrically?

Consider an $n \times n \times n$ cube C composed of $1 \times 1 \times 1$ cubes situated in the first octant with one corner at $(0, 0, 0)$ and its edges aligned with the coordinate axes. Consider a $k \times k \times k$ sub-cube with its edges aligned with the coordinate axes composed of the little $1 \times 1 \times 1$ cubes. We wish to count the total number of possible sub-cubes. Note that each sub-cube can be identified with its corner which is closest to the origin. If that corner is at (x, y, z), then we see that we simply need to count the number of triples of integers (x, y, z) such that $0 \le x, y, z \le n - k$. There are $n - k + 1$ possible choices for x, y and z independently so there are $(n - k + 1)^3$ such triples. Therefore

$$1^3 + 2^3 + \cdots + n^3 = \sum_{k=1}^{n} (n - k + 1)^3$$

gives the total number of such sub-cubes contained in C.

(b) Prove that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

for all $n \in \mathbb{N}$.

Proof. We use induction. Note that $\frac{(1)^2(1+1)^2}{4} = 1 = 1^3$, so the base step holds.
Now suppose that for some $k \in \mathbb{N}$ that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

Then

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2}{4}[k^2 + 4(k+1)] \\ &= \frac{(k+1)^2(k+1+1)^2}{4}. \end{aligned}$$

This verifies the inductive step.
We conclude by induction that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

for all $n \in \mathbb{N}$. □

Extra exercises

5.41 Show that there exists a rational number a and an irrational number b such that a^b is irrational.

Proof. Let $a = 2$ and $b = \frac{1}{2\sqrt{2}}$. Since $\sqrt{2}$ is irrational, it is easy to prove that b is irrational. Consider the number $x = 2^{\frac{1}{2\sqrt{2}}}$. If x is irrational, then the proof is complete. So suppose that x is rational. Now note that

$$x^{\sqrt{2}} = (2^{\frac{1}{2\sqrt{2}}})^{\sqrt{2}} = 2^{1/2} = \sqrt{2};$$

since we are assuming x is rational, and since we know that $\sqrt{2}$ we have found a rational number, x, and an irrational number, $\sqrt{2}$, such that $x^{\sqrt{2}}$ is irrational. □

5.49 Disprove the statement: There exist odd integers a and b such that $4 \mid 3a^2 + 7b^2$.
Disproof. We must disprove this statement by contradiction. Suppose that a and b are odd integers such that $4 \mid 3a^2 + 7b^2$.

Write $a = 2k+1$ and $b = 2l+1$ for some integers k and l. Then

$$\begin{aligned} 3a^2 + 7b^2 &= 3(2k+1)^2 + 7(2l+1)^2 \\ &= 3(4k^2 + 4k + 1) + 7(4l^2 + 4l + 1) \\ &= 4(3k^2 + 3k + 7l^2 + 7l + 2) + 2; \end{aligned}$$

shows that $4 \nmid 3a^2 + 7b^2$, which contradicts our assumption.

6.11 Prove that $\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n+2)(n+3)} = \frac{n}{3(n+3)}$ for every positive integer n.

Proof. We use a proof by mathematical induction. For the base step we observe that

$$\frac{1}{3\cdot 4} = \frac{1}{12} = \frac{1}{3(1+3)}.$$

For the inductive step, assume that k is a positive integer and that

$$\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(k+2)(k+3)} = \frac{k}{3(k+3)}.$$

Then we compute

$$\begin{aligned} \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+2)(k+3)} + \frac{1}{((k+1)+2)((k+1)+3)} &= \frac{k}{3(k+3)} + \frac{1}{(k+3)(k+4)} \\ &= \frac{k(k+4)+3}{3(k+3)(k+4)} \\ &= \frac{k^2+4k+3}{3(k+3)(k+4)} \\ &= \frac{(k+1)(k+3)}{3(k+3)(k+4)} \\ &= \frac{k+1}{3((k+1)+3)}, \end{aligned}$$

and this verifies the inductive step.

We conclude using the principle of mathematical induction that the statement holds for all positive integers n. □