Math 240: Introduction to Mathematical Thought

Homework 6, Solutions

Assigned exercises

5.2 Disprove the statement: If $n \in \{0, 1, 2, 3, 4\}$, then $2^n + 3^n + n(n-1)(n-2)$ is prime.

A counterexample to this statement is $n = 4$ since

$$2^4 + 3^4 + 4(4-1)(4-2) = 121 = 11^2$$

is not prime.

5.14 Prove that if $a \geq 2$ and b are integers, then $a \nmid b$ or $a \nmid (b+1)$.

Proof. We will prove this using the method of proof by contradiction. Assume that $a \geq 2$ and b are integers, and assume that $a|b$ and $a|(b+1)$. Then, by definition, $b = aj$ and $b + 1 = ak$ for some integers j and k. Therefore we have

$$aj + 1 = ak.$$

Rearranging we see that

$$1 = a(k - j).$$

Since $k - j$ is an integer we know from the last equation that $a|1$. However this is a contradiction since $a \geq 2$ and so a cannot divide 1. $\square$

5.20 Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.

Proof. Suppose, to obtain a contradiction, that $\sqrt{2} + \sqrt{3} = r$ for some nonzero $r \in \mathbb{Q}$. Then $\sqrt{2} = r - \sqrt{3}$. Squaring both sides of the last equation yields

$$2 = r^2 - 2r\sqrt{3} + 3.$$

Solving this equation for $\sqrt{3}$ we get

$$\sqrt{3} = \frac{1 + r^2}{2r}.$$

Note that the right hand side of the last equation is a rational number. We have reached a contradiction since we proved in class that $\sqrt{3}$ is irrational. $\square$

Extra exercises

5.13 Use proof by contradiction to prove that if a and b are odd integers, then $4 \nmid (a^2 + b^2)$.

Proof. We use a proof by contradiction. Suppose that a and b are odd integers, and assume that $4|(a^2+b^2)$. Write $a = 2k+1$ and $b = 2l+1$ for some integers k and l. Then

$$a^2+b^2 = (2k+1)^2+(2l+1)^2 = 4k^2+4k+1+4l^2+4l+1 = 4(k^2+k+l^2+l)+2.$$

Since k^2+k+l^2+l is an integer, this shows that $4 \not| (a^2+b^2)$, which is contrary to our assumption. $\square$

5.22 Let $S = \{p+q\sqrt{2} \mid p, q \in \mathbb{Q}\}$ and $T = \{r+s\sqrt{3} \mid r, s \in \mathbb{Q}\}$. Prove that $S \cap T = \mathbb{Q}$.

Proof. We start by observing that $\mathbb{Q}$ is a subset of both S and T. Hence $\mathbb{Q} \subseteq S \cap T$.

To finish the proof we need to show $S \cap T \subseteq \mathbb{Q}$. Let $x \in S \cap T$. Then there are $p, q, r, s \in \mathbb{Q}$ such that

$$x = p + q\sqrt{2} = r + s\sqrt{3}.$$

We will show that $qs = 0$. From this it follows that $q = 0$ or $s = 0$, and so $x = p$ or $x = r$, respectively, which shows $x \in \mathbb{Q}$. We argue by contradiction: suppose that $qs \neq 0$. Rearranging the last displayed equation yields

$$p - r = s\sqrt{3} - q\sqrt{2}.$$

Squaring we get

$$(p-r)^2 = 3s^2 - 2sq\sqrt{6} + 2q^2.$$

Since $sq \neq 0$ we can solve for $\sqrt{6}$ and we have

$$\sqrt{6} = \frac{(p-r)^2 - 3s^2 - 2q^2}{-2sq}.$$

This is a contradiction: $\sqrt{6}$ is an irrational number, but

$$\frac{(p-r)^2 - 3s^2 - 2q^2}{-2sq}$$

is a rational number since p, q, r, s are rational. We conclude that $qs = 0$, and the result follows.

$\square$

5.26 Prove that the sum of the squares of two odd integers cannot be the square of an integer.

Proof. Let x and y be odd integers. Then $x = 2k+1$ and $y = 2l+1$ for some $k, l \in \mathbb{Z}$. Then

$$x^2+y^2 = (2k+1)^2+(2l+1)^2 = 4k^2+4k+1+4l^2+4l+1 = 4(k^2+k+l^2+l)+2.$$

This shows that $x^2 + y^2 \equiv 2 \pmod 4$. Now let $z \in \mathbb{Z}$. Then z is congruent to either $0, 1, 2$ or 3 modulo 4. Note that $0^2 \equiv 0 \pmod 4$, $1^2 \equiv 1 \pmod 4$, $2^2 \equiv 0 \pmod 4$, $3^2 \equiv 1 \pmod 4$. Therefore z^2 is **not** congruent to 2 modulo 4. Therefore we cannot have $x^2 + y^2$ equal to z^2, as desired.

$\square$