Math 240: Introduction to Mathematical Thought

Homework 5, Solutions

Assigned exercises

4.48 Let $A = \{n \in \mathbb{Z} \ : \ 2|n\}$ and $B = \{n \in \mathbb{Z} \ : \ 4|n\}$. Let $n \in \mathbb{Z}$. Prove that $n \in A - B$ if and only if $n = 2k$ for some odd integer k.

Proof. We will prove the "only if direction" using a direct proof. Assume that $n \in \mathbb{Z}$ and $n \in A - B$. Since $n \in A$, we know that $2|n$, i.e., we know that $n = 2k$ for some integer k. If k was even, then we would have $k = 2m$ for some integer m, and from that it would follow that $n = 2k = 2(2m) = 4m$. The last set of equations imply that $4|n$, but we have assumed that $n \notin B$, so we know $4 \not| n$. We conclude that k is an odd integer, as was to be proved.

For the "if direction" we also employ a direct proof. Suppose that $n = 2k$ for some odd integer k. First of all note that $2|n$, so we know that $n \in A$. Second, write $k = 2l + 1$ for some integer l. Then

$$n = 2k = 2(2l + 1) = 4l + 2.$$

As discussed in class, this set of equations tells us that $4 \not| n$. Hence $n \notin B$. We conclude that $n \in A - B$, as desired. $\square$

4.56 Let A, B and C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

Proof. Notice that the three sets:

$$A - B, \quad A - C, \quad A - (B \cap C)$$

involved in the equation

$$(A - B) \cup (A - C) = A - (B \cap C)$$

are subsets of the set A. We put $U = A$, and consider A as the universal set. With this assumption we have $A - B = \overline{B}$, $A - C = \overline{C}$ and $A - (B \cap C) = \overline{B \cap C}$. The statement to be proved then is written as

$$\overline{B} \cup \overline{C} = \overline{B \cap C},$$

but this is one of De Morgan's Laws, which we know to be true. $\square$

4.68 Let A, B, C and D be sets. Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. We prove this statement by using a chain of bi-conditionals. (Note to the student reader: please make sure you understand why each bi-conditional holds.)

We have

$$\begin{aligned}(x,y) \in (A \times B) \cap (C \times D) &\iff (x,y) \in A \times B \text{ and } (x,y) \in C \times D \\ &\iff x \in A \cap C \text{ and } y \in B \cap D \\ &\iff (x,y) \in (A \cap C) \times (B \cap D);\end{aligned}$$

this proves that

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

$\square$

Extra exercises

4.42 Let A and B be sets. Prove that $A \cap B = A$ if and only if $A \subseteq B$.

Proof. First assume that $A \cap B = A$. We will prove that $A \subseteq B$. Let $a \in A$ be fixed, but arbitrary. Since we are assuming $A \cap B = A$, we know that $a \in A \cap B$. This means that $a \in B$. So we have proved $A \subseteq B$.

Now suppose that $A \subseteq B$. We will prove that $A \cap B = A$. It is always true that $A \cap B \subseteq A$, so we only need to show that $A \subseteq A \cap B$. Let $a' \in A$. We are assuming that $A \subseteq B$, so we know that $a' \in B$. Now we know that a' is in both A and B, i.e., $a' \in A \cap B$. So we have proved that $A \subseteq A \cap B$, and therefore that $A = A \cap B$. $\square$

4.53 Prove one of the distributive laws, namely, prove that for every three sets A, B, and C that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. To set notation, let $L = A \cap (B \cup C)$ and $R = (A \cap B) \cup (A \cap C)$.

First we show $L \subseteq R$. Let $x \in L$ be fixed, but arbitrary. By the definition of intersection of sets we know that $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, we have two cases to consider: $x \in B$ or $x \in C$. If $x \in B$, then we know that $x \in A \cap B$. Hence by the definition of the union of sets we know that $x \in (A \cap B) \cup (A \cap C)$, i.e., $x \in R$. The case in which $x \in C$ is

parallel, and we again see that $x \in R$. We have proved that $L \subseteq R$.

Last we show $R \subseteq L$. Let $y \in R$ be arbitrary. Then $y \in (A \cap B) \cup (A \cap C)$, so by the definition of set union, we know that $y \in A \cap B$ or $y \in A \cap C$. We consider these two cases. If $y \in A \cap B$, then by definition of set intersection we know that $y \in A$ and $y \in B$. Then we see that $y \in B \cup C$. Hence $y \in A \cap (B \cup C)$. The second case where $y \in A \cap C$ is parallel, and the conclusion is the same: $y \in A \cap (B \cup C)$. We have proved that $R \subseteq L$. □

4.65 Let A, B and C be **nonempty** sets. Prove that

$$A \times C \subseteq B \times C \text{ if and only if } A \subseteq B.$$

Proof. We give a direct proof of the "only if direction". Suppose that $A \times C \subseteq B \times C$. In order to show $A \subseteq B$, we choose a fixed but arbitrary $a \in A$. Using the assumption that C is nonempty, choose an element $c \in C$. Now consider the ordered pair (a, c); this ordered pair is in $A \times C$ by our choices, and moreover, since $A \times C \subseteq B \times C$, we also see that $(a, c) \in B \times C$. From this we can then say that $a \in B$. We have proved $A \subseteq B$.

For the converse, we also give a direct proof. Assume that $A \subseteq B$. We now show that $A \times C \subseteq B \times C$. Let $(a, c) \in A \times C$ be fixed, but arbitrary. Then we know that $a \in A$ and $c \in C$. Using the assumption that $A \subseteq B$, we know that $a \in B$. Therefore we can see that $(a, c) \in A \times C$. We have proved that $A \times C \subseteq B \times C$.

□

Note: In the last problem, only the assumption that C was nonempty was used. The hypothesis that A and B are nonempty is not needed.