Math 240: Introduction to Mathematical Thought

Homework 3, Solutions

Assigned exercises

3.16 Let $x \in \mathbb{Z}$. Prove that if $7x + 5$ is odd, then x is even.

Proof. We use a proof by contrapositive. Assume that x is an odd integer. Then $x = 2k + 1$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} 7x + 5 &= 7(2k + 1) + 5 \\ &= 14k + 12 \\ &= 2(7k + 6); \end{aligned}$$

since $7k + 6$ is an integer, by definition, we know that $7x + 5$ is an even integer. This is what we desired to prove. $\square$

3.24 Let $n \in \mathbb{Z}$. Prove that $2n^2 + n$ is odd if and only if $\cos \frac{n\pi}{2}$ is even.

Proof. We start by noting that the integer values of the cosine function come from the set $\{-1, 0, 1\}$. In particular, the only *even* integer value of the cosine function is zero. Moreover, standard facts about the cosine function imply that $\cos x = 0$ if and only if $x = \frac{n\pi}{2}$, where n is an arbitrary *odd* integer. This proves that $\cos \frac{n\pi}{2}$ is even if and only if n is odd. Therefore to prove the original statement, it suffices to prove:

$2n^2 + n$ is odd if and only if n is odd.

First suppose that n is odd. Then $n = 2m + 1$ for some integer m. So

$$2n^2+n = 2(2m+1)^2+(2m+1) = 2(4m^2+4m+1)+2m+1 = 2(4m^2+5m+1)+1.$$

Since $4m^2 + 5m + 1$ is an integer, we see that $2n^2 + n$ is odd, as desired.

Conversely, assume that n is even. Then $n = 2p$ for some integer p. Then

$$2n^2 + n = 2(2p)^2 + 2p = 8p^2 + 2p = 2(4p^2 + p).$$

Since $4p^2+p$ is an integer, we see that $2n^2+n$ is even, as desired. $\square$

3.26 Prove that if $n \in \mathbb{Z}$, then $n^2 - 3n + 9$ is odd.

Proof. We will use a proof by cases.
Case 1: n is even
Then $n = 2k$ for some integer k. So

$$n^2 - 3n + 9 = (2k)^2 - 3(2k) + 9 = 4k^2 - 6k + 9 = 2(2k^2 - 3k + 4) + 1.$$

Since $2k^2 - 3k + 4$ is an integer, we know that $n^2 - 3n + 9$ is odd, as desired.
Case 2: n is odd
Then $n = 2l + 1$ for some integer l. So

$$n^2 - 3n + 9 = (2l+1)^2 - 3(2l+1) + 9 = 4l^2 + 4l + 1 - 6l - 3 + 9 = 2(2l^2 - l + 3) + 1.$$

Since $2l^2 - l + 3$ is an integer, we know that $n^2 - 3n + 9$ is odd, as desired. □

Extra exercises

3.18 Let $x \in \mathbb{Z}$. Prove that $5x - 11$ is even if and only if x is odd.

Proof. For the "if direction", we start by assuming that x is odd. Then $x = 2k + 1$ for some integer k. So

$$5x - 11 = 5(2k + 1) - 11 = 10k - 6 = 2(5k - 3).$$

Since $5k - 3$ is an integer, we see that $5x - 11$ is even, as desired.
For the "only if direction", we use a proof by contrapositive. Suppose that x is even. Then $x = 2l$ for some integer l. So

$$5x - 11 = 5(2l) - 11 = 10l - 11 = 2(5l - 6) + 1.$$

Since $5l - 6$ is an integer, we see that $5x - 11$ is odd, as desired. □

3.21 Let $n \in \mathbb{Z}$. Prove that $(n + 1)^2 - 1$ is even if and only if n is even.

Proof. For the "only if direction" we use a proof by contrapositive. Suppose that n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then

$$(n+1)^2 - 1 = (2k+1+1)^2 - 1 = (2k+2)^2 - 1 = 4k^2 + 8k + 4 - 1 = 2(2k^2 + 4k + 1) + 1.$$

Since $2k^2 + 4k + 1 \in \mathbb{Z}$, we know that $(n + 1)^2 - 1$ is odd, as was to be proved.
For the "if direction", we use a direct proof. Assume that n is even. Then $n = 2l$ for some $l \in \mathbb{Z}$. Then

$$(n + 1)^2 - 1 = (2l + 1)^2 - 1 = 4l^2 + 4l + 1 - 1 = 2(2l^2 + 2l).$$

Since $2l^2+2l \in \mathbb{Z}$, we know that $(n+1)^2-1$ is even, as desired. □

3.27 Prove that if $n \in \mathbb{Z}$, then $n^3 - n$ is even.

Proof. We use a proof by cases.
Case 1: n is even
Then $n = 2k$ for some $k \in \mathbb{Z}$. So

$$n^3 - n = (2k)^3 - 2k = 8k^3 - 2k = 2(4k^3 - k).$$

Since $4k^3-k$ is an integer, we see that n^3-n is even, as desired.
Case 2: n is odd
Then $n = 2l + 1$ for some $l \in \mathbb{Z}$. So

$$n^3-n = (2k+1)^3-2k-1 = 8k^3+12k^2+6k+1-2k-1 = 2(4k^3+6k^2+2k).$$

Since $4k^3 + 6k^2 + 2k$ is an integer, we see that $n^3 - n$ is even, as desired.

□