Math 240: Introduction to Mathematical Thought

Homework 2, Solutions

1.40 (a) $\bigcup_{i=1}^{5} A_{2i} = \{1, 3, 5, 7, 9, 11\}$
(b) $\bigcup_{i=1}^{5} (A_i \cap A_{i+1}) = \emptyset$
(c) $\bigcup_{i=1}^{5} (A_{2i-1} \cap A_{2i+1}) = \{2, 4, 6, 8, 10\}$

1.60 For $A = \{\emptyset, \{\emptyset\}\}$, determine $A \times \mathcal{P}(A)$.

Solution: We have $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, A\}$. So

$$A \times \mathcal{P}(A) = \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\emptyset, \{\{\emptyset\}\}), (\emptyset, A), (\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\emptyset\}, \{\{\emptyset\}\}), (\{\emptyset\}, A)\}.$$

1.62 $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 4\}$

1.66 The set $(A \times B) \cup (B \times A)$ consists of all of the points on the boundary of the square of side length 2 with vertices: $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$.

3.10 Prove that if a and c are odd integers, then $ab + bc$ is even for every integer b.

Proof. Assume that a and c are odd integers. Then we may write $a = 2k + 1$ and $b = 2l + 1$ for some integers k and l. Let b be an arbitrary integer. Then

$$\begin{aligned} ab + bc &= b(a + c) \\ &= b(2k + 1 + 2l + 1) \\ &= b(2k + 2l + 2) \\ &= 2b(k + l + 1); \end{aligned}$$

since $b(k + l + 1)$ is an integer, we conclude that $ab + bc$ is an even integer, as desired. □

3.11 Let $n \in \mathbb{Z}$. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

Proof. We start by assuming that $n \in \mathbb{Z}$ and $1 - n^2 > 0$. By inspection, the only solution of this inequality is $n = 0$. So $3n - 2$ is the integer $3(0) - 2 = -2$. We know that -2 is even. □

3.15 Let $A = \{n \in \mathbb{Z} \mid n > 2, \text{ and } n \text{ is odd}\}$ and $B = \{n \in \mathbb{Z} \mid n < 11\}$. Prove that if $n \in A \cap B$, then $n^2 - 2$ is prime.

Proof. First of all, it is easy to check that $A \cap B = \{3, 5, 7, 9\}$. By inspection, when $n = 3, 5, 7, 9$, the values of $n^2 - 2$ are $7, 23, 47, 79$, respectively. Since the last list of numbers consists of numbers that are all prime, we conclude the result. $\square$