Let $F$ denote the additive group of all functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let $K$ be the subgroup of $F$ consisting of the constant functions. Find a subgroup of $F$ to which $F/K$ is isomorphic.

Let $H = \{g \in F \mid g(0) = 0\}$. Define a function $\Phi : F \rightarrow F$ by $\Phi(f) = f - f(0)$ where $(f - f(0))(x) = f(x) - f(0)$ for all $x \in \mathbb{R}$.

Claim: The group $F/K$ is isomorphic to the subgroup $H$.

Proof. We will apply the Fundamental Homomorphism Theorem to the function $\Phi$.

First of all let $f, h \in F$. Then

$$
\Phi(f + h) = (f - f(0)) + (h - h(0))
= (f + h) - (f + h)(0)
= \Phi(f) + \Phi(h),
$$

shows $\Phi$ is a homomorphism.

Second of all,

$f$ is a constant function iff $f(x) = f(0)$ for all $x \in \mathbb{R}$
iff $f(x) - f(0) = 0$ for all $x \in \mathbb{R}$
iff $(f - f(0))(x) = 0$ for all $x \in \mathbb{R}$
iff $f - f(0) = 0$
iff $\Phi(f) = 0$.

Therefore $\text{Ker}(\Phi) = K$.

Thirdly we claim the image of $\Phi$ is $H$. Note that $\Phi(f)(0) = f(0) - f(0) = 0$ so that $\Phi(f) \in H$. On the other hand, let $g \in H$ so that $g(0) = 0$. Then $\Phi(g) = g - g(0) = g$, which shows that $g \in \phi[F]$. Hence $\Phi[F] = H$ as claimed.

Finally, we apply the Fundamental Homomorphism Theorem to conclude $F/K$ is isomorphic to the subgroup $H$. □
Show that if $G$ is nonabelian, then the factor group $G/Z(G)$ is not cyclic.

**Proof.** We will prove the contrapositive statement. We assume that $G/Z(G)$ is a cyclic group. Let $xZ(G)$ be a generator of $G/Z(G)$ for some $x \in G$. Let $a, b \in G$.

Consider the cosets $aZ(G), bZ(G)$. Then there are integers $m$ and $n$ such that $aZ(G) = (xZ(G))^m = x^mZ(G)$ and $bZ(G) = (xZ(G))^n = x^nZ(G)$. In particular $a = x^mz_1$ and $b = x^nz_2$ for some $z_1, z_2 \in Z$.

Now, using the fact that $z_1, z_2$ commute with all group elements, we have

$$ab = x^mz_1x^nz_2$$
$$= x^mx^nz_1z_2$$
$$= x^{m+n}z_2z_1$$
$$= x^n z_2 x^m z_1$$
$$= x^n z_2 x^m z_1$$
$$= ba.$$ 

Therefore $G$ is abelian. $\square$

Mark each of the following as true or false.

(a) Every factor group of a cyclic group is cyclic.

This statement is **true**. The homomorphic image of a cyclic group is cyclic.

(b) A factor group of a noncyclic group is again noncyclic.

This statement is **false**. Consider $S_3$ and its normal subgroup $\langle (1, 2, 3) \rangle$. We know $S_3$ is not cyclic. The group $S_3/\langle (1, 2, 3) \rangle$ is isomorphic to $\mathbb{Z}_2$, a cyclic group.

(c) $\mathbb{R}/\mathbb{Z}$ under addition has no element of order 2.

This statement is **false**. The coset $\frac{1}{2} + \mathbb{Z}$ has order 2.

(d) $\mathbb{R}/\mathbb{Z}$ has elements of order $n$ for all $n \in \mathbb{Z}^+$.

This statement is **true**. The coset $\frac{1}{n} + \mathbb{Z}$ is easily checked to have order $n$.

(e) $\mathbb{R}/\mathbb{Z}$ under addition has an infinite number of elements of order 4.

This statement is **false**. Since $\mathbb{R}/\mathbb{Z}$ is isomorphic to $U$, the unit circle under multiplication, we know that $\mathbb{R}/\mathbb{Z}$ has exactly two elements of order 4.

Give an example of a group $G$ having no elements of finite order $> 1$ but having a factor group $G/H$, all of whose elements are of finite order.

Let $G$ be the group $\mathbb{Z}$ under addition. Then $G$ has no elements of finite order $> 1$. Let $H = 16\mathbb{Z}$. Then all elements of $G/H = \mathbb{Z}_{16}$ have finite order.
Let $\phi : G \rightarrow G'$ be a group homomorphism, and let $N$ be a normal subgroup of $G$. Show that $\phi[N]$ is a normal subgroup of $\phi[G]$.

Proof. Let $\phi : G \rightarrow G'$ be a group homomorphism, and let $N$ be a normal subgroup of $G$. Let $x \in \phi[G]$ and $a \in \phi[N]$. Then there exist $y \in G$ and $b \in N$ such that $x = \phi(y)$ and $a = \phi(b)$.

Since $N$ is normal we know $yby^{-1} \in N$. Furthermore

$$
\phi(yby^{-1}) = \phi(y)\phi(b)\phi(y)^{-1}
= xax^{-1}.
$$

The last equation shows $xax^{-1} \in \phi[N]$. We conclude that $\phi[N]$ is a normal subgroup of $\phi[G]$. □