1. (12 points)

Let $U = \{n \in \mathbb{N} \mid n \leq 20\}$ be the universal set. Define subsets of U as follows.

$$A = \{a \in U : \ a^2 \leq 16\}$$

$$B = \{b \in U : \ b \equiv 1 \,(\text{mod } 3)\}$$

$$C = \{c \in U : \ 10|c\}$$

In parts (a) - (f), determine the set by listing its elements between braces.

(a) A

$A = \{1, 2, 3, 4\}$

(b) B

$B = \{1, 4, 7, 10, 13, 16, 19\}$

(c) C

$C = \{10, 20\{$

(d) $A \cup B$

$A \cup B = \{1, 2, 3, 4, 7, 10, 13, 16, 19\{$

(e) $A \cap B$

$A \cap B = \{1, 4\}$

(f) $\overline{B}$

$\overline{B} = \{2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20\}.$

2. (4 points)

Let $A = \{1, 2\}$ and $B = \{\emptyset\}$. Determine the following sets by listing their elements between braces.

(a) $A \times B$

$A \times B = \{(1, \phi), (2, \phi)\}$

(b) $\mathcal{P}(A) \times \mathcal{P}(B)$

$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, A\}$

$\mathcal{P}(B) = \{\phi, B\}$

$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\phi, \phi), (\phi, B), (\{1\}, \phi), (\{1\}, B), (\{2\}, \phi), (\{2\}, B),$

$(A, \phi), (A, B)\}.$

3. (9 points)
Let $n \in \mathbb{Z}$.

(a) State the definition of the phrase: n **is even**.

An integer n is <u>even</u> if $n = 2k$ for some integer k.

(b) **Prove:** If n is even, then n^3 is even.

<u>Proof</u> Assume that $n \in \mathbb{Z}$ is even. Write $n = 2k$, $k \in \mathbb{Z}$.

Then $n^3 = (2k)^3 = 8k^3 = 2(4k^3)$. Since $4k^3 \in \mathbb{Z}$, we know that n^3 is even. $\boxtimes$

(c) **Prove:** If n^3 is even, then n is even.

<u>Proof</u> We prove the contrapositive.

Assume that n is odd. Then $n = 2k+1$, $k \in \mathbb{Z}$.

So $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1$ (using Pascal's triangle)

$$= 2(4k^3 + 6k^2 + 3k) + 1.$$

Since $4k^3 + 6k^2 + 3k \in \mathbb{Z}$ we know that n^3 is odd. $\boxtimes$

4. (6 points)

Let $x \in \mathbb{R}$.

(a) State the definition of the phrase: x **is a rational number**.

A real number x is a rational number if there exist $a, b \in \mathbb{Z}$ such that

$$x = \frac{a}{b}.$$

(b) **Prove:** $\sqrt[3]{2}$ is an irrational number.

Proof Suppose, to the contrary, that $\sqrt[3]{2}$ is a rational number. Let $a, b \in \mathbb{Z}$ such that $\sqrt[3]{2} = \frac{a}{b}$. Moreover, we assume that $\frac{a}{b}$ is reduced; that is: a and b have no common factors other than -1 and 1.

Now $2 = \frac{a^3}{b^3}$, so $2b^3 = a^3$. Therefore a^3 is even. By the previous problem, a is even. Write $a = 2k$, $k \in \mathbb{Z}$. Then $2b^3 = (2k)^3 = 8k^3$, so $b^3 = 4k^3 = 2(2k^3)$. Hence b^3 is even, and it follows that b is even. We have reached a contradiction since we have proved that a and b have a common factor of 2, but this is contrary to our assumption. $\boxtimes$

5. (6 points)

Let $a, b \in \mathbb{Z}$, with $a \neq 0$.

(a) State the definition of the phrase: a **divides** b.

For integers a, b with $a \neq 0$, a divides b means

$b = ak$ for some integer k.

(b) **Prove:** If $3|ab$, then $3|a$ or $3|b$.

Proof: We start by rewriting the statement as:

If $ab \equiv 0 \pmod{3}$, then $a \equiv 0 \pmod{3}$ or $b \equiv 0 \pmod{3}$.

We employ a proof by contrapositive.

Assume that $a \not\equiv 0 \pmod{3}$ and $b \not\equiv 0 \pmod{3}$. Then, without loss of generality, there are three cases.

Case 1 $a \equiv 1 \pmod{3}$, $b \equiv 1 \pmod{3}$

Then $ab \equiv (1)(1) \equiv 1 \pmod{3}$, so $ab \not\equiv 0 \pmod{3}$.

Case 2 $a \equiv 1 \pmod{3}$, $b \equiv 2 \pmod{3}$.

Then $ab \equiv (1)(2) \pmod{3} \equiv 2 \pmod{3}$, so $ab \not\equiv 0 \pmod{3}$.

Case 3 $a \equiv 2 \pmod{3}$, $b \equiv 2 \pmod{3}$

Then $ab \equiv (2)(2) \equiv 4 \equiv 1 \pmod{3}$, so $ab \not\equiv 0 \pmod{3}$. ▨

6. (6 points)

Let $a, b, m \in \mathbb{Z}$, with $m \geq 2$.

(a) State the definition of the phrase: a **is congruent to** b **modulo** m.

For integers a, b, m with $m \geq 2$,

a is congruent to b modulo m means that

$$m \mid a - b.$$

(b) **Prove:** For every integer n, $n^3 \equiv 4n \pmod 3$.

Proof: We consider three cases. Let $n \in \mathbb{Z}$.

Case 1 $\quad n \equiv 0 \pmod 3$

Then $n^3 \equiv 0^3 \equiv 0 \pmod 3$, and $4n \equiv 4(0) \equiv 0 \pmod 3$,

so $n^3 \equiv 4n \pmod 3$.

Case 2 $\quad n \equiv 1 \pmod 3$

Then $n^3 \equiv 1^3 \equiv 1 \pmod 3$, and $4n \equiv 4(1) \equiv 4 \equiv 1 \pmod 3$,

so $n^3 \equiv 4n \pmod 3$.

Case 3 $\quad n \equiv 2 \pmod 3$

Then $n^3 \equiv (2)^3 \equiv 8 \equiv 2 \pmod 3$, and $4n \equiv 4(2) \equiv 8 \equiv 2 \pmod 3$,

So $n^3 \equiv 4n \pmod 3$. ∎

7. (7 points)

Choose **one** of the following. If you attempt both, **clearly indicate which one you would like me to grade.**

(a) **Prove:** For all sets A and B,

$$B = (B - A) \cup (A \cap B).$$

(b) **Prove:** For every positive integer n,

$$1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n.$$

Proof of (a)

Let $b \in B$. Then either $b \in A$ or $b \notin A$.

If $b \in A$, then $b \in A \cap B$, so $b \in (B-A) \cup (A \cap B)$.

If $b \notin A$, then $b \in B - A$, so $b \in (B-A) \cup (A \cap B)$.

Hence $B \subseteq (B-A) \cup (A \cap B)$.

Next, let $c \in (B-A) \cup (A \cap B)$. So $c \in B - A$ or $c \in A \cap B$. In either case we know that $c \in B$.

Hence $(B-A) \cup (A \cap B) \subseteq B$. ▨

Proof of (b)

For the base step, observe that $4(1)-3 = 1$, and $1 = 2(1)^2 - 1$.

For the inductive step assume that for some $k \in \mathbb{N}$ that $1 + 5 + 9 + \cdots + (4k-3) = 2k^2 - k$. Then

$$\begin{aligned} 1 + 5 + \cdots + (4k-3) + (4(k+1)-3) &= 2k^2 - k + 4k + 1, \text{ by assumption} \\ &= 2k^2 + 3k + 1 \\ &= 2(k+1)^2 - (k+1). \end{aligned}$$

▨