Problem 1 (5 points)

Let $f(x) = -3x^2 + 12x - 11$. Parts (a)-(d) below refer to this function.

(a) Write $f(x)$ in the form $f(x) = a(x-h)^2 + k$.

$f(x) = -3(x^2 - 4x) - 11 = -3(x^2 - 4x + 4) - 11 + 12$

$\boxed{f(x) = -3(x-2)^2 + 1}$

(b) State the vertex of the associated parabola.

$\boxed{\text{vertex} = (2, 1)}$

(c) Determine all of the x-intercepts. Leave your answer in **exact form**.

$-3(x-2)^2 + 1 = 0 \Rightarrow -3(x-2)^2 = -1$

$\Rightarrow (x-2)^2 = \frac{1}{3} \Rightarrow x - 2 = \pm\sqrt{\frac{1}{3}} \Rightarrow \boxed{x = 2 \pm \frac{1}{\sqrt{3}}}$

x-intercepts $\boxed{\left(2 + \frac{1}{\sqrt{3}}, 0\right), \left(2 - \frac{1}{\sqrt{3}}, 0\right)}$

(d) Determine the range of f. Write your answer in interval notation.

$\boxed{\text{range}(f) = (-\infty, 1]}$

Problem 2 (5 points)

Let $f(x) = (x-1)(x+2)^2$. Parts (a)-(c) below refer to this function.

(a) Determine all of the zeros of f and state their multiplicities.

zeros	1	-2
multiplicity	1	2

(b) Complete the following sentences.

As $x \to -\infty$, $f(x) \to$ $-\infty$.

As $x \to \infty$, $f(x) \to$ ∞.

(c) Determine the intervals on which $f(x)$ is positive.

Interval	Sign of $f(x)$
~~$(-2,\infty)$~~	
$(-\infty,-2)$	negative
$(-2,1)$	negative
$(1,\infty)$	positive

$\boxed{(1,\infty)}$

Problem 3 (5 points)

Let $p(x) = x^5 - x^4 + x^2 - 3x + 1$ and $d(x) = x - 2$. Parts (a)-(b) below refer to these functions.

(a) Use synthetic division or long division to find the **quotient** and **remainder** if $p(x)$ is divided by $d(x)$.

$$\begin{array}{r|rrrrrr} 2 & 1 & -1 & 0 & 1 & -3 & 1 \\ & & 2 & 2 & 4 & 10 & 14 \\ \hline & 1 & 1 & 2 & 5 & 7 & 15 \end{array}$$

quotient: $\boxed{q(x) = x^4 + x^3 + 2x^2 + 5x + 7}$

remainder: $\boxed{r(x) = 15}$

(b) Use the Rational Zero Theorem to determine if $p(x)$ has any **rational** zeros. Show all of your work.

RZT $\Rightarrow$ possible rational zeros are ± 1

$p(1) = -1$, $p(-1) = -1 - 1 + 1 + 3 + 1 = 3$

Therefore $\boxed{p(x) \text{ has } \underline{\text{no}} \text{ rational zeros}}$

Problem 4 (5 points)

Let $p(x) = -x^4 + 2x^3 - 7x - 4$.
Use Descartes' Rule of Signs to determine the **possible** number of **negative** zeros of p. Show your work. You do **not** need to determine the number of **actual** negative zeros.

$$p(-x) = -(-x)^4 + 2(-x)^3 - 7(-x) - 4$$

$$p(-x) = -x^4 - 2x^3 + 7x - 4$$

variation of sign of $p(-x)$ is 2

possible number of negative zeros: $\boxed{2 \text{ or } 0}$

Problem 5 (5 points)

Let $x = -i$ and $y = 1 + \sqrt{-4}$. Write each of the following complex numbers in the form $a + bi$ for some real numbers a and b.

$y = 1 + 2i$

(a) $x + y$

$-i + 1 + 2i = \boxed{1 + i}$

(b) $x - y$

$-i - (1 + 2i) = \boxed{-1 - 3i}$

(c) xy

$-i(1 + 2i) = -i - 2i^2 = -i + 2$

$\boxed{2 - i}$

(d) $\frac{x}{y}$

$\frac{-i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{-i + 2i^2}{1 - 4i^2} = \frac{-2 - i}{5}$

$\boxed{\frac{-2}{5} - \frac{1}{5} i}$

Problem 6 (5 points)

Let $f(x) = \dfrac{x-2}{x^2-2x-3}$. Parts (a)-(d) refer to this function.

(a) Determine the domain of f. Write your answer in interval notation.

$x^2-2x-3 = (x-3)(x+1) = 0 \Rightarrow x=3$ or $x=-1$

domain $(f) = \boxed{(-\infty,-1) \cup (-1,3) \cup (3,\infty)}$

(b) Determine all of the vertical asymptotes.

$\boxed{x=3 \text{ and } x=-1}$

(c) Determine the horizontal asymptote.

$\boxed{y=0}$

(d) Determine all of the x-intercepts.

$$\frac{x-2}{x^2-2x-3} = 0 \Rightarrow x-2=0 \Rightarrow x=2$$

x-intercept: $\boxed{(2,0)}$

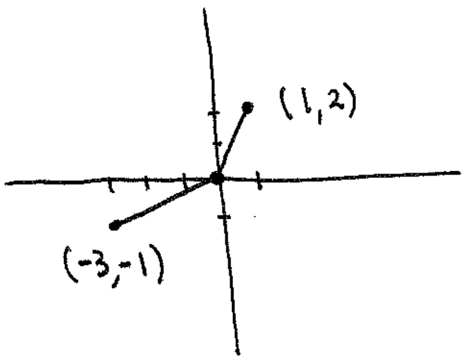
Problem 7 (5 points)

Let

$$f(x) = \begin{cases} \left(\frac{1}{3}\right)x & -3 \le x \le 0 \\ 2x & 0 < x \le 1 \end{cases}$$

Parts (a)-(b) below refer to this function.

(a) Sketch the graph of f and explain why the graph shows that f is one-to-one. What test are you applying?

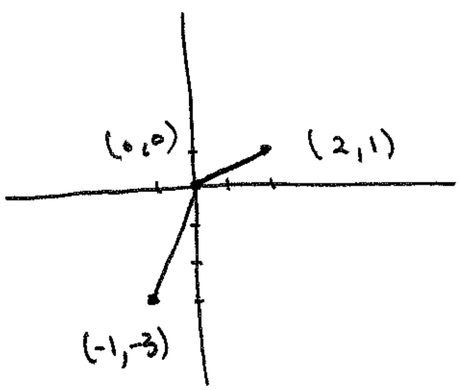


The graph passes the horizontal line test. So f is one-to-one.

(b) Sketch the graph of f^{-1}.

points on the graph of f^{-1}:

(-1,-3), (0,0), (2,1)



Problem 8 (5 points)

Plutonium is a radioactive element that has a half-life of 24,360 years. Suppose the initial amount of plutonium in a sample is 10 pounds.

(a) Find an exponential function of the form $A(t) = A_0 e^{kt}$ that gives the amount of plutonium left in the sample after t years.

$$5 = 10e^{k(24,360)} \Rightarrow \frac{1}{2} = e^{24,360k} \Rightarrow \ln\left(\frac{1}{2}\right) = 24,360k$$

$$\Rightarrow k = \frac{\ln(0.5)}{24,360} \quad ; \quad A_0 = 10 \qquad k \approx -2.8 \times 10^{-5}$$

$$\boxed{A(t) = 10\, e^{\left(\frac{\ln(0.5)}{24,360}\right)t}}$$

(b) How much plutonium remains after 10,000 years? Round your answer to the nearest tenth of a pound.

$$A(10000) = 10\, e^{\left(\frac{\ln(0.5)}{24,360}\right)(10,000)}$$

$$\approx \boxed{7.5 \text{ pounds}}$$

Problem 9 (5 points)

Solve the following logarithmic equation. Do not forget to check your solutions.

$$\log_5(x) = 1 - \log_5(x-4)$$

$\log_5 x + \log_5 (x-4) = 1$

$\log_5 (x(x-4)) = 1$

$x(x-4) = 5$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5$ or $x = -1$

<u>note</u> $x = -1$ is not in the domain of $\log_5 (x)$.

$\boxed{x = 5}$

Problem 10 (5 points)

(a) Write the following logarithm as a difference of logarithmic expressions. Eliminate all exponents and radicals, and evaluate logarithms whenever possible.

$$\ln\left(\frac{\sqrt[3]{x^2}}{e^2}\right)$$

$$\ln\left(\frac{x^{2/3}}{e^2}\right) = \ln\left(x^{2/3}\right) - \ln\left(e^2\right)$$

$$= \boxed{\frac{2}{3}\ln(x) - 2}$$

(b) Write the following expression as the logarithm of a single expression. Hint: First write 2 as $\log(a)$ for some a.

$$3\log(x) + 2$$

$$2 = \log(100)$$

$$3\log x + \log 100 = \log x^3 + \log 100$$

$$= \boxed{\log(100x^3)}$$