Session I Linear Mapping Figures

We begin our introduction to mapping figures by a consideration of linear functions:

\[ y = f(x) = mx + b \]
RULES FOR FUNCTIONS

YOUR FRIENDLY NEIGHBORHOOD FUNCTION CONSISTS OF TWO SETS
AND A BUNCH OF ARROWS THAT CHEESE.

RULE 1

THE ARROWS ALWAYS START FROM THE SAME SET CALLED THE

DOMAIN AND GO TO THE OTHER SET, CALLED THE

RANGE.

EVERYTHING IN THE DOMAIN RULE SET MUST HAVE EXACTLY ONE

ARROW FROM IT. EVERYTHING IN THE RANGE SET MUST HAVE AT

LEAST ONE ARROW TO IT.

So two or more arrows can hit the same thing in the range-set, but

nobody ever arrow two arrows from any particular thing in the domain-set.

Using arrows in the RULES unfortunately has its drawbacks—functions become more elaborate,

the arrows can get pretty difficult
to follow . . .

In the study hall we all got about the

rubbers. There are usually many

rubbers around, or so it seems.

There are so many that the guys

make no difference. However, the

rubbers themselves are no

rubbers from the rubber-set. To

rubber-set is just a group of

rubbers.

The exercise is much of their own

make. We need to make sure

two steps depend on creating groups,

function.

By the graph we know you have to

create two groups. We will do just

one step.

Here we can see that the

rubbers are all different parts of the

rubber-set. However, only one

rubber is necessary in this case.

This means we need to create only

one group. That is, the domain-set

will just be the

rubbers.

Remember that the domain of any

function is always part of the

domain-set. We need to make

sure that it is the correct one.

The domain is always the one

where there are no arrows from

rubbers-set in the domain-set.

We need to create two groups.

In the graph oh 1.5, 9 will be just

one arrow.
Mapping Diagrams and Functions

- **Function Diagrams**, by Henri Picciotto
  Excellent Resources!
  - Henri Picciotto's Math Education Page
  - Some rights reserved
- Flashman, Yanosko, Kim
  https://www.math.duke.edu/education/prep02/teams/prep-12/

Outline of Remainder of Morning...

- Linear Functions: They are everywhere!
- Tables
- Graphs
- Mapping Figures
- Excel, Winplot and other technology Examples
- Characteristics and Questions
- Understanding Linear Functions Visually.
Visualizing Linear Functions

• Linear functions are both necessary, and understandable—even without considering their graphs.
• There is a sensible way to visualize them using "mapping figures."
• Examples of important function features (like "slope" and intercepts) will be illustrated with mapping figures.
• Examples of activities for students that engage understanding both function and linearity concepts.
• Examples of these mappings using simple straight edges as well as technology such as Winplot (freeware from Peanut Software), Geogebra, and possibly Mathematica and GSP.
• Winplot is available from http://math.exeter.edu/rparris/peanut/

Linear Functions: They are everywhere!

• Where do you find Linear Functions?
  - At home:
  - On the road:
  - At the store:
  - In Sports/ Games

Linear Functions: Tables

Complete the table.

- For which \( x \) is \( f(x) > 0 \)?
Connect Points (x, 5x - 7):

<table>
<thead>
<tr>
<th>X</th>
<th>5x - 7</th>
</tr>
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<tbody>
<tr>
<td>3</td>
<td>8</td>
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<td>2</td>
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Connect the Points (x, 5x - 7):

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• Connect point x to point 5x - 7 on axes.

What happens before the graph.

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Linear Functions: Mapping Figures
What happens before the graph.

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Function-Equation Questions with mapping figures

- Solving a linear equations: 2x + 1 = 5
  \[ 2x + 1 = x + 2 \]
  - \( f(x) = 2x + 1 \): For which \( x \) does \( f(x) = 5 \)?
  - \( g(x) = x + 2 \): For which \( x \) does \( f(x) = g(x) \)?

- Find “fixed points” of \( f \): \( f(x) = 2x + 1 \)
  - For which \( x \) does \( f(x) = x \)?

Simple Examples are important!
- \( f(x) = x + C \): Added value: \( C \)
- \( f(x) = mx \): Scalar Multiple: \( m \)

Interpretations of \( m \):
  - slope
  - rate
  - Magnification factor
  - \( m > 0 \): Increasing function
  - \( m = 0 \): Constant function [WS Example]
  - \( m < 0 \): Decreasing function [WS Example]

Simple Examples are important!
\( f(x) = mx + b \) with a mapping figure
Five examples:
- Example 1: \( m = -2; b = 1 \): \( f(x) = -2x + 1 \)
- Example 2: \( m = 2; b = 1 \): \( f(x) = 2x + 1 \)
- Example 3: \( m = \frac{1}{2}; b = 1 \): \( f(x) = \frac{1}{2}x + 1 \)
- Example 4: \( m = 0; b = 1 \): \( f(x) = 0x + 1 \)
- Example 5: \( m = 1; b = 1 \): \( f(x) = x + 1 \)
Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

**Example 1:** $m = -2; b = 1$

- Each arrow passes through a single point, which is labeled $F = \{-2,1\}$.
  - The point $F$ completely determines the function $f$.
  - Given a point/number, $x$, on the source line,
  - There is a unique arrow passing through $F$.
  - Meeting the target line at a unique point/number, $-2x + 1$,
  - Which corresponds to the linear function's value for the point/number, $x$.

**Example 2:** $m = 2; b = 1$

- Each arrow passes through a single point, which is labeled $F = \{2,1\}$.
  - The point $F$ completely determines the function $f$.
  - Given a point/number, $x$, on the source line,
  - There is a unique arrow passing through $F$.
  - Meeting the target line at a unique point/number, $2x + 1$,
  - Which corresponds to the linear function's value for the point/number, $x$.

**Example 3:** $m = 1/2; b = 1$

- Each arrow passes through a single point, which is labeled $F = \{1/2,1\}$.
  - The point $F$ completely determines the function $f$.
  - Given a point/number, $x$, on the source line,
  - There is a unique arrow passing through $F$.
  - Meeting the target line at a unique point/number, $1/2x + 1$,
  - Which corresponds to the linear function's value for the point/number, $x$.

**Example 4:** $m = 0; b = 1$

- Each arrow passes through a single point, which is labeled $F = \{0,1\}$.
  - The point $F$ completely determines the function $f$.
  - Given a point/number, $x$, on the source line,
  - There is a unique arrow passing through $F$.
  - Meeting the target line at a unique point/number, $0x + 1$,
  - Which corresponds to the linear function's value for the point/number, $x$. 
Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples

Example 5: $m = 1; b = 1$

$f(x) = x + 1$

- Unlike the previous examples, in this case it is not a single point that determines the mapping figure, but the single arrow from 0 to 1, which we designate as $F[1,1]$.
- It can also be shown that this single arrow completely determines the function. Thus, given a point/number, $x$, on the source line, there is a unique arrow passing through a parallel to $F[1,1]$ meeting the target line at a unique point/number, $x + 1$, which corresponds to the linear function’s value for the point/number, $x$.

- The single arrow completely determines the function $f$.
  - Given a point/number, $x$, on the source line.
  - There is a unique arrow through $x$ parallel to $F[1,1]$.
  - Meeting the target line at a unique point/number, $x + 1$, which corresponds to the linear function’s value for the point/number, $x$.

End of Session I

- Questions
- Break - food and thought
- Partner/group integration task

Function-Equation Questions

with linear focus points

- Solve a linear equations:
  - $2x + 1 = 5$
  - $2x + 1 = -x + 2$
- Use focus to find $x$.

- “fixed points” : $f(x) = x$
- Use focus to find $x$.

Morning and Lunch Break: Think about These Problems (in Groups 1-2; 3-4)

M.1 How would you use the Linear Focus to find the mapping figure for the function inverse for a linear function when $m \neq 0$?

M.2 How does the choice of axis scales affect the position of the linear function focus point and its use in solving equations?

M.3 Describe the visual features of the mapping figure for the quadratic function $f(x) = x^2$. How does this generalize for even functions where $f(x) = f(-x)$?

M.4 Describe the visual features of the mapping figure for the cubic function $f(x) = x^3$. How does this generalize for odd functions where $f(x) = -f(x)$?
Session II More on Linear Mapping Figures

We continue our introduction to mapping figures by a consideration of the composition of linear functions.

Compositions are keys!

An example of composition with mapping figures of simpler (linear) functions.
- $g(x) = 2x; \ h(u)=u+1$
- $f(x) = h(g(x)) = h(u)$
  where $u = g(x) = 2x$
- $f(x) = (2x) + 1 = 2x + 1$
  $f(0) = 1$  slope = 2

Linear Functions can be understood and visualized as compositions with mapping figures of simpler linear functions.
- $f(x) = 2x + 1 = (2x) + 1$
  - $g(x) = 2x; \ h(u)=u+1$
  - $f(0) = 1$  slope = 2

Example: $f(x) = 2(x-1) + 3$
- $g(x)=x-1$  $h(u)=2u; \ k(t)=t+3$
  - $f(1)= 3$  slope = 2
Inverses, Equations and Mapping Figures

- Inverse: If \( f(x) = y \) then \( inv_f(y) = x \).
- So to find \( inv_f(b) \) we need to find any and all \( x \) that solve the equation \( f(x) = b \).
- How is this visualized on a mapping figure?
- Find \( b \) on the target axis, then trace back on any and all arrows that "hit" \( b \).

Mapping Figures and Inverses

Inverse linear functions:

- Use transparency for mapping figures:
  - Copy mapping figure of \( g \) to transparency.
  - Flip the transparency to see mapping figure of inverse function \( g \). ("before or after")
  \( inv_g(a) = a \); \( g(inv_g(b)) = b \).
- Example i: \( g(x) = 2x \); \( inv_g(x) = \frac{1}{2} x \)
- Example ii: \( h(x) = x + 1 \); \( inv_h(x) = x - 1 \)
End of Session II

- Questions
- Lunch Break - food and thought
- Partner/group integration task

Lunch Break: Think about These Problems (in Groups 1-3; 4-5)

L.1 Describe the visual features of the mapping figure for the quadratic function \( f(x) = x^2 \).

L.2 Describe the visual features of the mapping figure for the quadratic function \( f(x) = A(x-h)^2 + k \) using composition with simple linear functions.

L.3 Describe the visual features of a mapping figure for the square root function \( g(x) = \sqrt{x} \) and relate them to those of the quadratic \( f(x) = x^2 \).

L.4 Describe the visual features of the mapping figure for the reciprocal function \( f(x) = \frac{1}{x} \).
   Domain? Range? “Asymptotes” and “infinity”? Function Inverse? 

L.5 Describe the visual features of the mapping figure for the linear fractional function \( f(x) = \frac{A}{x-h} + k \) using composition with simple linear functions.
   Domain? Range? “Asymptotes” and “infinity”? Function Inverse? 

Session III More on Mapping Figures: Quadratic, Exponential and Logarithmic Functions

We continue our introduction to mapping figures by a consideration of quadratic, exponential and logarithmic functions.

Examples on Excel, Winplot, Geogebra

- Excel example: 
- Winplot examples:
  - Linear Mapping examples
- Geogebra examples:
  - dynagraphs.ggb
  - Composition
Web links

- https://www.math.duke.edu/education/prep02/teams/prep-12/
- http://users.humboldt.edu/flashman/TFLINX.HTM
- http://demonstrations.wolfram.com/ComposingFunctionsUsingDynagraphs/

Quadratic Functions

- Usually considered as a key example of the power of analytic geometry— the merger of algebra with geometry.
- The algebra of this study focuses on two distinct representations of these functions which mapping figures can visualize effectively to illuminate key features.
  - $f(x) = Ax^2 + Bx + C$
  - $f(x) = A (x-h)^2 + k$

Examples

- Use compositions to visualize
  - $f(x) = 2 (x-1)^2 = 2x^2 - 4x + 2$
  - $g(x) = 2 (x-1)^2 + 3 = 2x^2 - 4x + 5$
- Observe how even symmetry is transformed.
- These examples illustrate how a mapping figure visualization of composition with linear functions can assist in understanding other functions.

Quadratic Mapping Figures

$f(x) = 2 (x-1)^2 = 2x^2 - 4x + 2$
Quadratic Mapping Figures

\[ g(x) = 2(x-1)^2 + 3 = 2x^2 - 4x + 5 \]

Quadratic Equations and Mapping Figures

- To solve \( f(x) = Ax^2 + Bx + C = 0 \).
- Find 0 on the target axis, then trace back on any and all arrows that “hit” 0.
- Notice how this connects to \( x = \frac{-B}{2A} \) for symmetry and the issue of the number of solutions.

Definition

- Algebra Definition
  \( b^L = N \) if and only if \( \log_b(N) = L \)
- Functions:
  \( f(x) = b^x = y; \quad \text{inv}f(y) = \log_b(y) = x \)
  \( \log_b = \text{inv}f \)
“Simple” Applications

I invest $1000 @ 3% compounded continuously. How long must I wait till my investment has a value of $1500? Solution: $A(t) = 1000 \ e^{0.03t}$.

Find $t$ where $A(t) = 1500$.

Visualize this with a mapping figure before further algebra.

Solution: $A(t) = 1000 \ e^{0.03t}$.

Find $t$ where $A(t) = 1500$.

Consider simpler mapping figure on next slide.
Simple Applications

Solution: \( A(t) = 1000 e^{0.03t} \).

Find \( t \) where \( A(t) = 1500 \).

Algebra: Find \( t \) where

\( u=0.03t \) and \( 1.5 = e^u \).

Consider simpler mapping figure and solve with logarithm:

\( u=0.03t = \ln(1.5) \) and

\( t = \frac{\ln(1.5)}{0.03} \approx 13.52 \)


\( e^{x+y} = e^x e^y = u^y \) where \( u = e^x \) and \( y = e^x \)

Thus by definition:

\[ x = \ln y ; t = \ln u; \]

And \( t+x = \ln(u^y) \),

SO

\[ \ln u + \ln y = \ln(u^y) \]

End of Session III

• Questions
• Break - food and thought
• Partner/group integration task
Session IV More on Mapping Figures: Trigonometry and Calculus Connections

We complete our introduction to mapping figures by a consideration of trigonometric functions and some connections to calculus.

Sine and cosine of $t$ measured on the vertical and horizontal axes.

Note the visualization of periodicity.

Tangent Interpreted on Unit Circle

- $\tan(t)$ measured on the axis tangent to the unit circle.

- Note the visualization of periodicity.

Even and odd on Mapping Figures

Even

Odd

$f(a) = f(-a)$

$f(a) = -f(a)$
An Even Function

\[ f(a) = f(-a) \]

An Odd Function

\[ f(-a) = -f(a) \]

Trigonometric functions and symmetry:

\[
\begin{align*}
\cos(-t) &= \cos(t) \quad \text{for all } t. \quad \text{EVEN} \\
\sin(-t) &= -\sin(t) \quad \text{for all } t. \quad \text{ODD} \\
\tan(-t) &= -\tan(t) \quad \text{for all } t. 
\end{align*}
\]

Justifications from unit circle mapping figures for sine, cosine and tangent.

Trig Equations and Mapping Figures

- To solve \( \text{trig}(x) = z \).
- Find \( z \) on the target axis, then trace back on any and all arrows that “hit” \( z \).
- Notice how this connects to periodic behavior of the trig functions and the issue of the number of solutions in an interval.
- This also connects to understanding the inverse trig functions.
Solving Simple Trig Equations:

Solve \( \text{trig}(t) = z \) from unit circle mapping figures for sine, cosine and tangent.

Winplot Examples for Trig Functions
Trig Linear Compositions

Compositions with Trig Functions

Example: \( y = f(x) = \sin(x) + 3 \)

- Mapping figure for \( y = f(x) = \sin(x) + 3 \) considered as a composition:
  - First: \( u = \sin(x) \)
  - Second: \( y = u + 3 \) so the result is
  - \( y = (\sin(x)) + 3 \)
Example: Graph of \( y = \sin(x) + 3 \)

Graph of \( y = \sin(x) + 3 \) [Winplot]

- Amplitude: 1
- Period: \( 2\pi \)

Example: \( y = f(x) = 3 \sin(x) \)

- Mapping figure for
- \( y = f(x) = 3 \sin(x) \) considered as a composition:
  - First: \( u = \sin(x) \)
  - Second: \( y = 3u \) so the result is
  - \( y = 3(\sin(x)) \)

Example: Graph of \( y = 3 \sin(x) \)

Graph of \( y = 3 \sin(x) \) [Winplot]

- Amplitude: 3
- Period: \( 2\pi \)

Interpretations

- \( y = 3\sin(x) \):
  - \( t \rightarrow (\cos(t), \sin(t)) \rightarrow (3\cos(t), 3\sin(t)) \)
    unit circle magnified to circle of radius 3.
- \( Y = \sin(x) + 3 \):
  - \( t \rightarrow (\cos(t), \sin(t)) \rightarrow (\cos(t), \sin(t) + 3) \)
    unit circle shifted up to unit circle with center \((0, 3)\).

Show with winplot: circles_sines.wp2;
Scale change before trig.
Mapping figures and graphs for \( f(x) = \sin(3x) \)
- Amplitude and period

Connection to solving equations:
- Example: \( \sin(2x) = 1 \);
  - \( 2x = \pi/2, 3\pi/2 \)
  - \( x = \pi/4, 3\pi/4 \)
  - Difference is period: \( (5\pi - \pi)/4 = \pi \).

Altogether!
- \( f(x) = 3 \sin(2x+\pi/3) + 2 \)
- Mapping figure: Before \( u = 2x+\pi/3 \)
  - After \( y = 3z + 2 \)
- MIDDLE: \( z = \sin(u) \).
- Amplitude:3, period: \( \pi \), and shift: ????.
- Visualize on circle. Dot races and mapping figures.
- Solve equations for period and shift.
- \( u = 0 \) and \( u = 2\pi \). Period = difference in \( x \).
Mapping figure

\[ f(x) = 3 \sin(2x + \pi/3) + 2 \]

Mapping figure:
- Before \( u = 2x + \pi/3 \)
- After \( y = 3x + 2 \)
- MIDDLE: \( z = \sin(u) \).

Graph

- \( f(x) = 3 \sin(2x + \pi/3) + 2 \)

Thanks

The End!

Questions?
flashman@humboldt.edu
http://www.humboldt.edu/~mef2

More References
More References


- "Dynagraphs)–helping students visualize function dependency • GeoGebra User Forum

- "degenerated" dynagraph game ("x" and "y" axes are superimposed) in GeoGebra: http://www.uff.br/cdme/c1d/c1d-html/c1d-en.html

More References

Thanks
The End! REALLY!

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