Using Mapping Diagrams to Understand Linear Functions

Martin Flashman

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Professor of Mathematics
Humboldt State University
flashman@humboldt.edu
http://users.humboldt.edu/flashman

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Links:

http://users.humboldt.edu/flashman/Presentations/UCDMP/UCDMP.MD.LINKS.html

Background Questions

- Are you familiar with Mapping Diagrams?
- Have you used Mapping Diagrams to teach functions?
- Have you used Mapping Diagrams to teach content besides function definitions?

Mapping Diagrams

A.k.a.
Function Diagrams
Dynagraphs

How Linear Functions Fit into Other Functions: Quadratic Example
Will be reviewed at end. ☺

$g(x) = 2\,(x-1)^2 + 3$

Steps for g:

1. **Linear:** Subtract 1.
2. **Square result.**
3. **Linear:** Multiply by 2 then add 3.

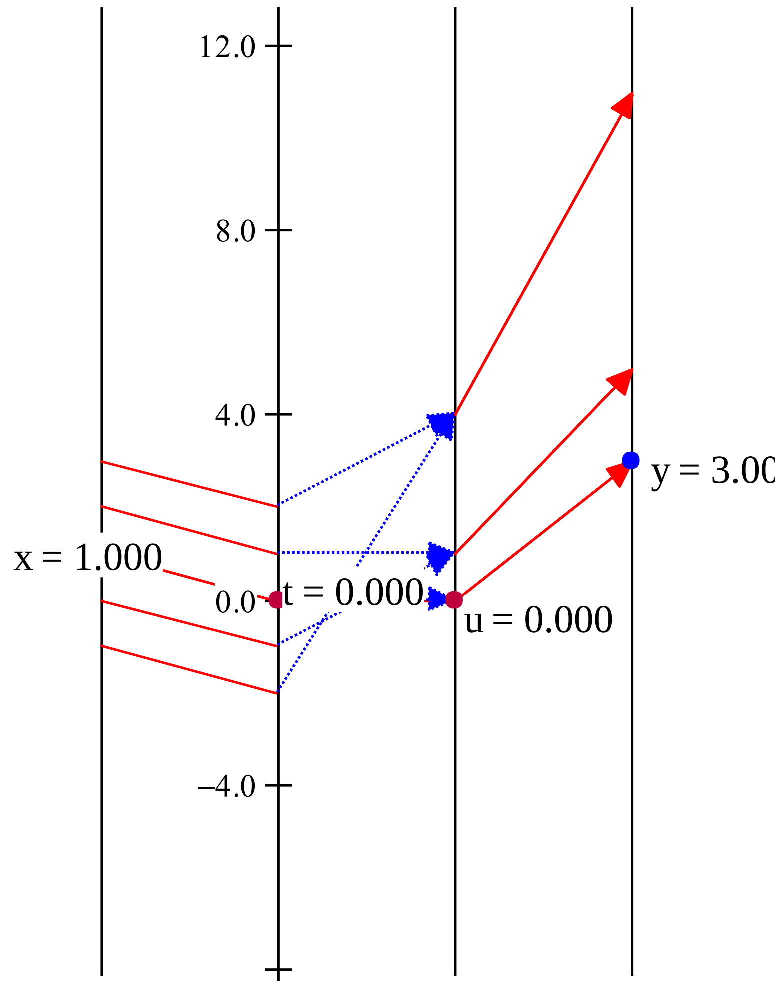
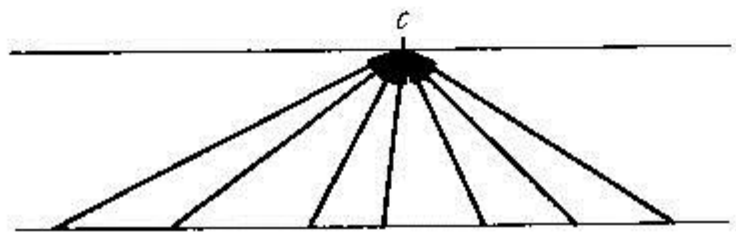
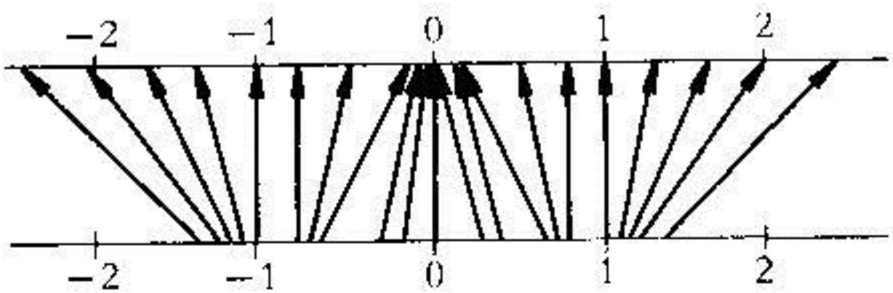


Figure from Ch. 5
Calculus by M. Spivak



(a) $f(x) = c$



(b) $f(x) = x^3$

FIGURE 2

Visualizing Linear Functions

- Linear functions are both necessary, and understandable- even without considering their graphs.
- There is a sensible way to visualize them using "mapping diagrams."
- Examples of important function features (like rate and intercepts) can be illustrated with mapping diagrams.
- Activities for students engage understanding for both function and linearity concepts.
- Mapping diagrams can use simple straight edges as well as technology.

Main Resource

- Mapping Diagrams from A(lgebra) B(asics) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource Book on Function Visualizations Using Mapping Diagrams (Preliminary Sections- NOT YET FOR publication)

- http://users.humboldt.edu/flashman/MD/section-1.1VF.html

Linear Mapping diagrams

We begin our more detailed introduction to mapping diagrams by a consideration of linear functions :

" $y = f(x) = mx + b$ "

Distribute Worksheet *now.*

Do Problem 1

Prob 1: Linear Functions -Tables

x	$5x - 7$
3	
2	
1	
0	
-1	
-2	
-3	

Complete the table.

$x = 3, 2, 1, 0, -1, -2, -3$

$f(x) = 5x - 7$

$f(0) =$ ____?

For which x is $f(x) > 0$?

Linear Functions: Tables

X	5 x – 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Complete the table.

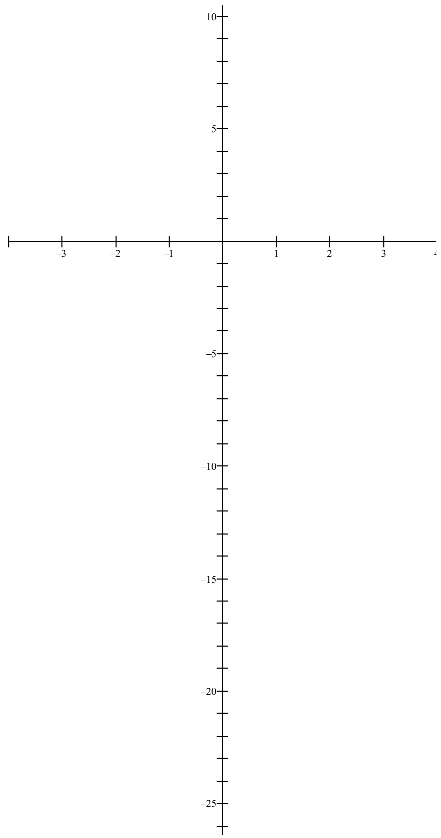
$x = 3,2,1,0,-1,-2,-3$

$f(x) = 5x - 7$

$f(0) = \_\_\_\_?$

For which x is $f(x) > 0$?

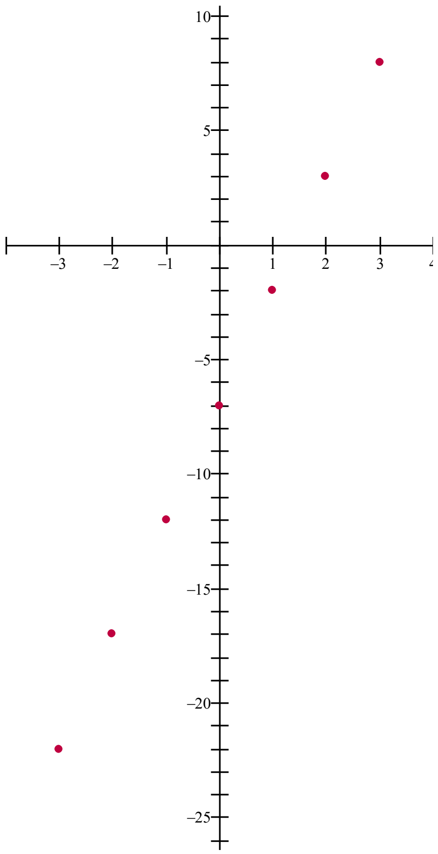
Linear Functions: On Graph



Plot Points (x , 5x - 7):

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

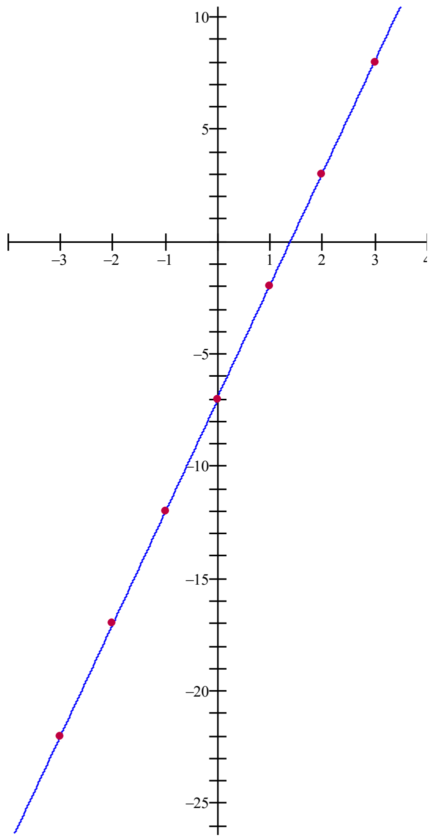
Linear Functions: On Graph



Connect Points
$(x , 5x - 7)$:

X	5 x – 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: On Graph



Connect the Points

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: Mapping diagrams Visualizing the table.

- Connect point x to point 5x – 7 on axes

X	5 x – 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: Mapping diagrams Visualizing the table.

- Connect point x to point 5x – 7 on axes

X	5 x – 7
3	8
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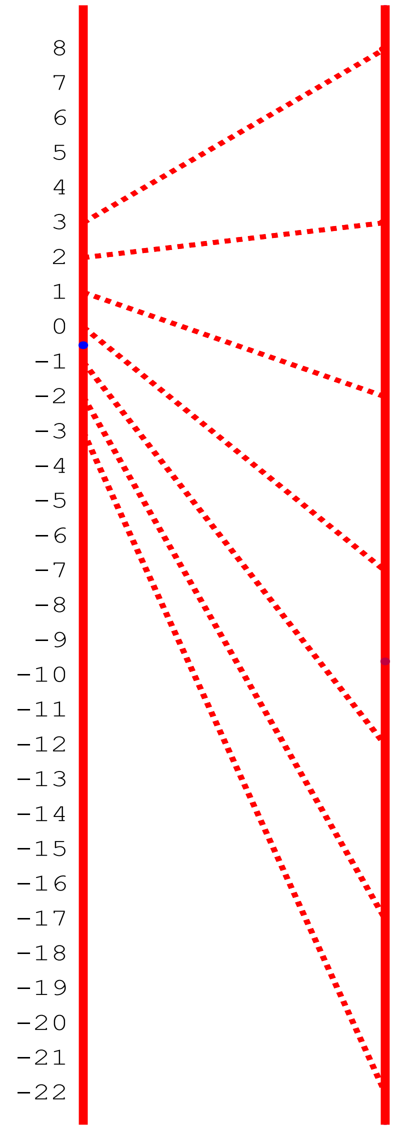


Technology Examples

- Excel example
- Geogebra example

Simple Examples are important!

- $f(x) = x + C$ Added value: C
- $f(x) = mx$ Scalar Multiple: m

  Interpretations of m:
  - slope
  - rate
  - Magnification factor
  - $m > 0$: Increasing function
  - $m < 0$: Decreasing function
  - $m = 0$: Constant function

Simple Examples are important!

$f(x) = mx + b$ with a mapping diagram --

Five examples:

Back to Worksheet Problem #2

- Example 1: $m = -2$; $b = 1$: $f(x) = -2x + 1$
- Example 2: $m = 2$; $b = 1$: $f(x) = 2x + 1$
- Example 3: $m = \frac{1}{2}$; $b = 1$: $f(x) = \frac{1}{2}x + 1$
- Example 4: $m = 0$; $b = 1$: $f(x) = 0x + 1$
- Example 5: $m = 1$; $b = 1$: $f(x) = x + 1$

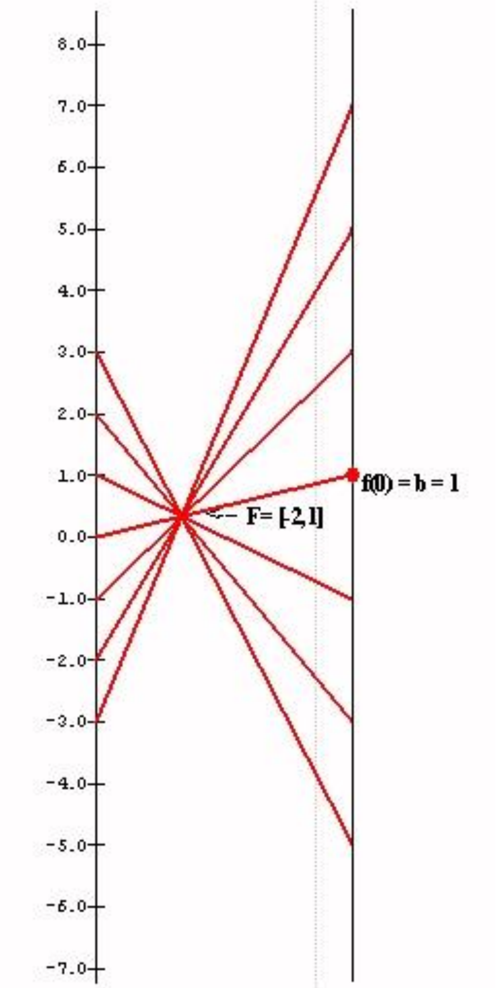
Visualizing $f(x) = \boldsymbol{mx} + \boldsymbol{b}$ with a mapping diagram -- Five examples:

Example 1: m = -2; b = 1

$f(x) = -2x + 1$

- Each arrow passes through a single point, which is labeled **F = [- 2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $-2x + 1$,

which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples:

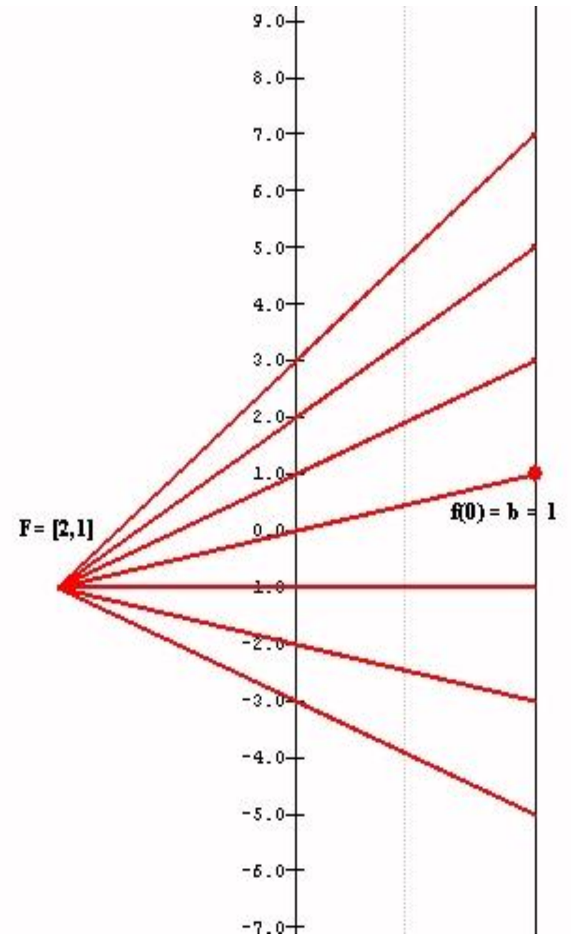
Example 2: $m = 2$; $b = 1$
$f(x) = 2x + 1$

Each arrow passes through a single point, which is labeled

F = [2,1].

- The point **F** completely determines the function *f*.
  - **given a point** / number, *x*, on the source line,
  - there is a **unique arrow passing through F**
  - **meeting** the target line at **a unique point** / number, $2x + 1$,

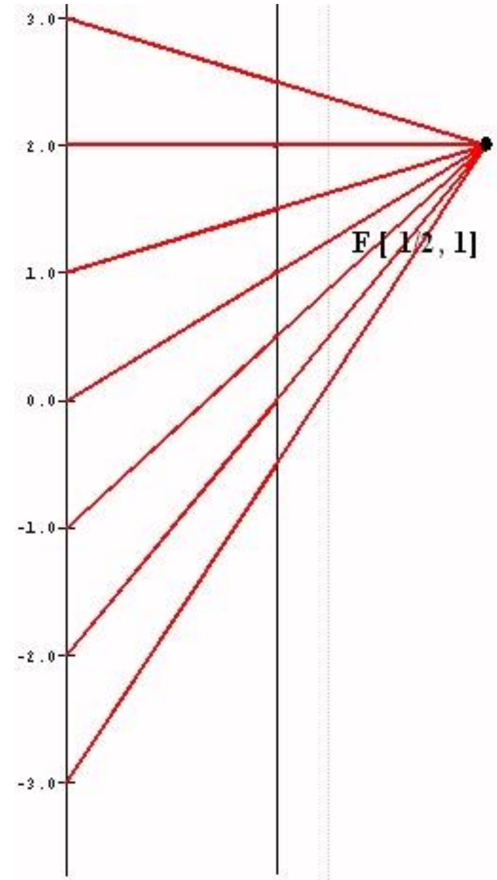
which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples:

- **Example 3: m = 1/2; b = 1**
  $f(x) = \frac{1}{2}x + 1$
- Each arrow passes through a single point, which is labeled **F = [1/2,1]**.
  - The point **F** completely determines the function f.
    - **given a point** / number, x, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $\frac{1}{2}x + 1$,

    which corresponds to the linear function's value for the point/number, x.

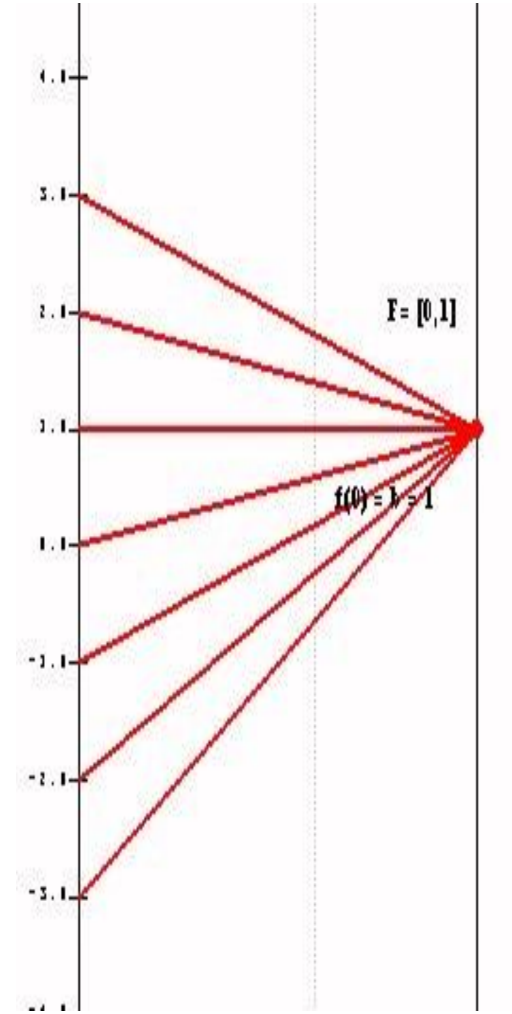


Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples:

- **Example 4:** $m = 0$; $b = 1$
  f(x) = 0 x + 1

- Each arrow passes through a single point, which is labeled **F = [0,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $f(x)=1$,

    which corresponds to the linear function's value for the point/number, *x*.



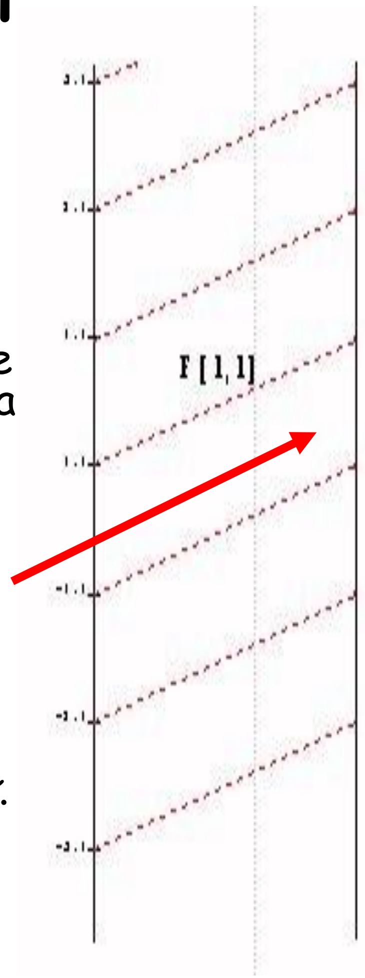
Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples

Example 5: $m = 1$; $b = 1$

$f(x) = x + 1$

- Unlike the previous examples, in this case it is not a single point that determines the mapping diagram, but the single arrow from 0 to 1, which we designate as F[1,1]
- It can also be shown that this single arrow completely determines the function. Thus, given a point / number, x, on the source line, there is a unique arrow passing through ***x parallel to*** F[1,1] meeting the target line a unique point / number, $x + 1$, which corresponds to the linear function's value for the point/number, x.
  - The single arrow completely determines the function f.
    - **given a point** / number, x, on the source line,
    - there is a **unique arrow** through ***x parallel to*** F[1,1]
    - **meeting** the target line at **a unique point** / number, $x + 1$,

which corresponds to the linear function's value for the point/number, x.



Function-Equation Questions

with linear focus points (Problem 3)

- Solve a linear equation:

  $2x+1 = 5$

  – Use focus to find x.

Function-Equation Questions

with linear focus points (Problem 4)

Suppose f is a linear function with $f(1) = 3$ and $f(3) = -1$.

- Without algebra
  - Use focus to find $f(0)$.
  - Use focus to find x where $f(x) = 0$.

More on Linear Mapping diagrams

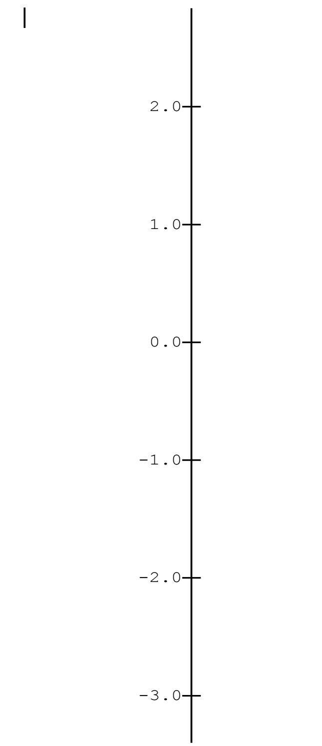
We continue our introduction to mapping diagrams by a consideration of the **composition of linear functions**.

Do Problem 5

Problem 5: Compositions are keys!

An example of composition with mapping diagrams of simpler (linear) functions.

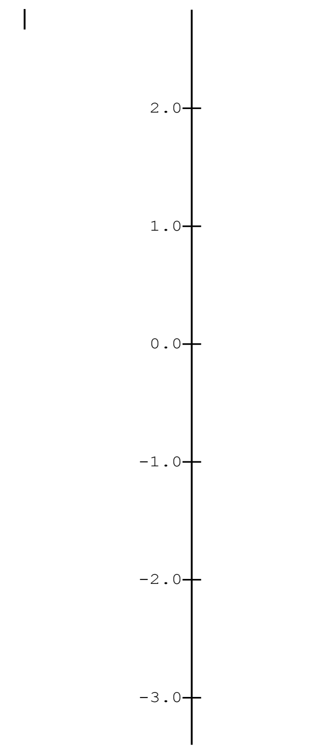
- $g(x) = 2x$; $h(x)=x+1$
- $f(x) = h(g(x)) = h(u)$
  where $u = g(x) = 2x$
- $f(x) = (2x) + 1 = 2x + 1$
  $f(0) = 1$ $m = 2$



Compositions are keys!

All Linear Functions can be understood and visualized as compositions with mapping diagrams of simpler linear functions.

- $f(x) = 2x + 1 = (2x) + 1$:
  - $g(x) = 2x$; $h(u)=u+1$
  - $f(0) = 1$ $m = 2$



Compositions are keys!

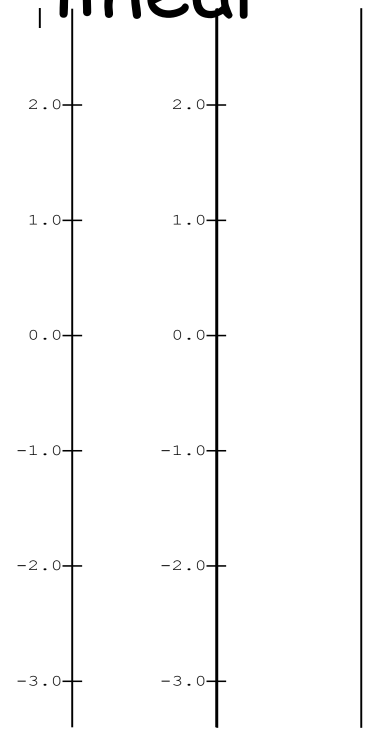
All Linear Functions can be understood and visualized as compositions with mapping diagrams of simpler linear functions.

Point Slope Example:

$f(x) = 2(x-1) + 3$

$g(x)=x-1$ $h(u)=2u$; $k(t)=t+3$

- $f(1)= 3$ slope = 2



Questions for Thought

- For which functions would mapping diagrams add to the understanding of composition?
- In what other contexts are composition with "x+h" relevant for understanding function identities?
- In what other contexts are composition with "-x" relevant for understanding function identities?

Inverses, Equations and Mapping diagrams

- Inverse: If $f(x) = y$ then $f^{-1}(y) = x$.
- So to find $f^{-1}(b)$ we need to find any and all x that solve the equation
$$f(x) = b.$$
- How is this visualized on a mapping diagram?
- Find b on the target axis, then trace back on any and all arrows that "hit" b.

Mapping diagrams and Inverses

Inverse linear functions:

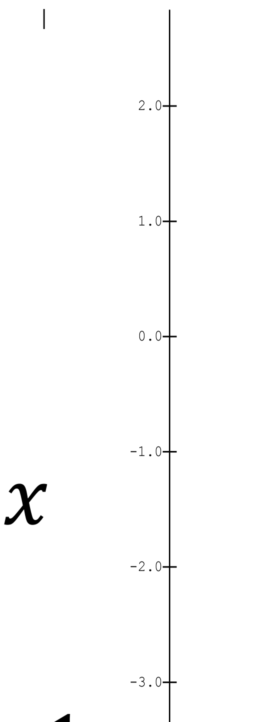
Classroom Activity

- Use transparency for mapping diagrams-
  - Copy mapping diagram of f to transparency.
  - Flip the transparency to see mapping

  diagram of inverse function $g = f^{-1}$.

  ("before or after")

$$f(g(b)) = b; \quad g(f(a)) = a$$



- Example i: $g(x) = 2x$; $g^{-1}(x) = \frac{1}{2}x$
- Example ii:

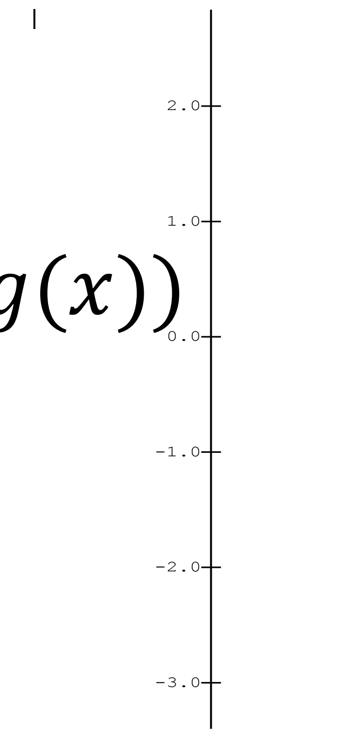
$$h(x) = x + 1\,; h^{-1}(x) = x - 1$$

Mapping diagrams and Inverses

Inverse linear functions:

- socks and shoes with mapping diagrams
- $g(x) = 2x$; $g^{-1}(x) = \frac{1}{2}x$
- $h(x) = x + 1$; $h^{-1}(x) = x - 1$
- $f(x) = 2x + 1 = (2x) + 1 = h(g(x))$
- $g(x) = 2x$; $h(u) = u + 1$
  - The inverse of f:

$$f^{-1}(x) = g^{-1}(h^{-1}(x)) = \frac{1}{2}(x - 1)$$

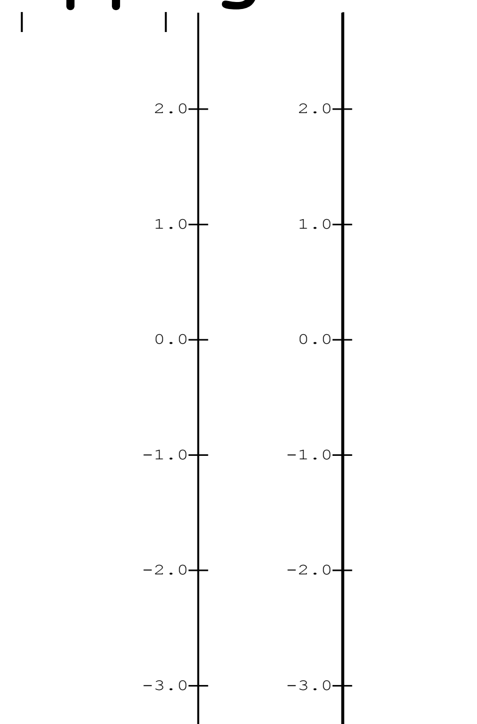


Mapping diagrams and Inverses

Inverse linear functions:

- "socks and shoes" with mapping diagrams
- $f(x) = 2(x-1) + 3$:
  - $g(x) = x - 1$
  - $h(u) = 2u$
  - $k(t) = t + 3$
  - The inverse of f:

$$f^{-1}(x) = \frac{1}{2}(x-3) + 1$$



Questions for Thought

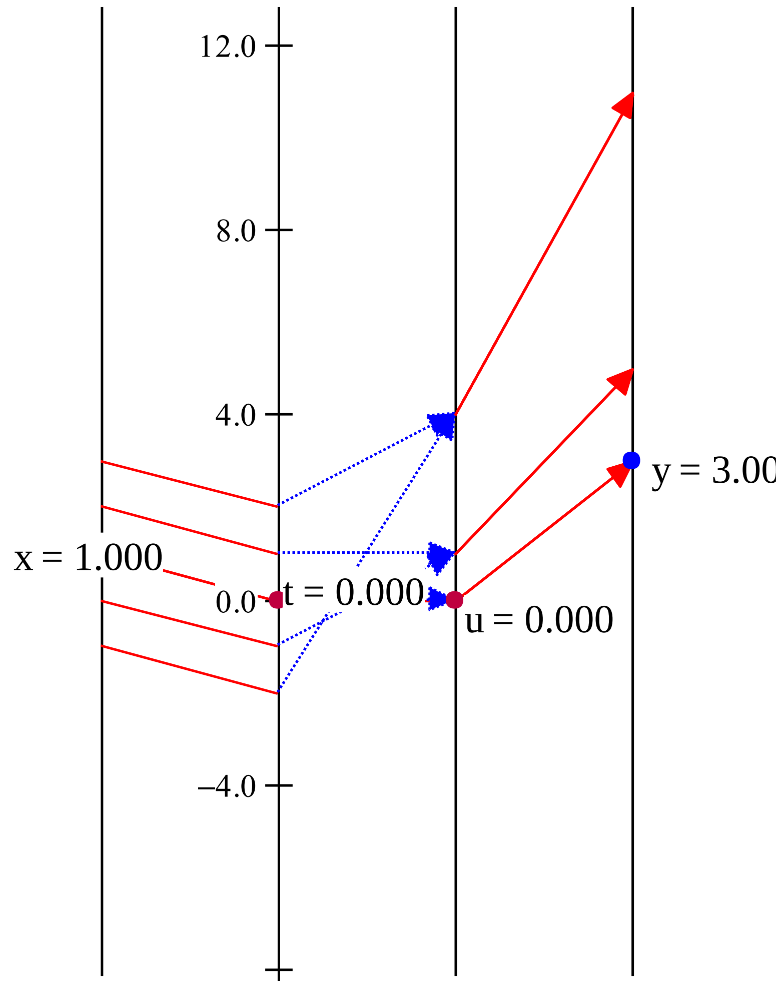
- For which functions would mapping diagrams add to the understanding of inverse functions?
- How does “socks and shoes” connect with solving equations and justifying identities?

Closer: Quadratic Example From Preface. ☺

$g(x) = 2\,(x-1)^2 + 3$

Steps for g:

1. **Linear:** Subtract 1.
2. **Square result.**
3. **Linear:** Multiply by 2 then add 3.



Thanks
The End!

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Questions?

flashman@humboldt.edu
http://users.humboldt.edu/flashman

References

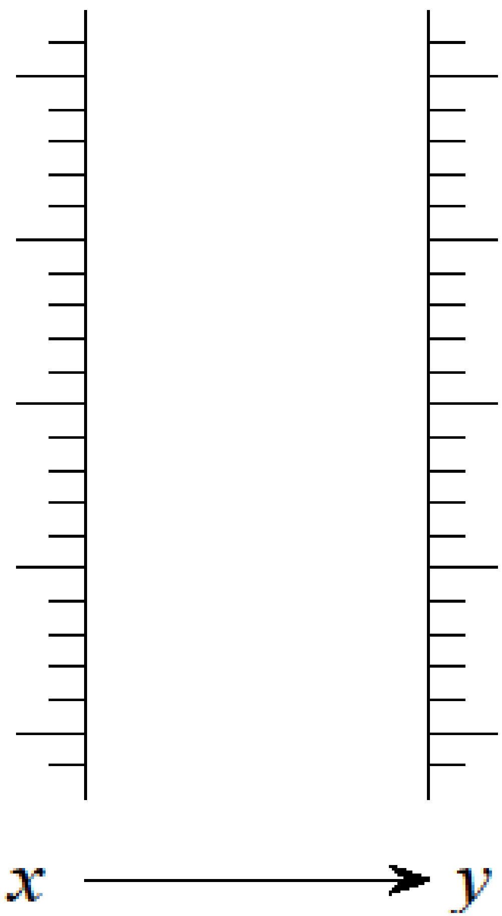
Mapping Diagrams and Functions

- SparkNotes › Math Study Guides › Algebra II: Functions Traditional treatment.
  - http://www.sparknotes.com/math/algebra2/functions/
- **Function Diagrams. by Henri Picciotto Excellent Resources!**
  - **Henri Picciotto's Math Education Page**
  - **Some rights reserved**
- Flashman, Yanosko, Kim https://www.math.duke.edu//education/prep02/teams/prep-12/

Function Diagrams by Henri Picciotto

Function Diagrams

Henri Picciotto, www.picciotto.org/math-ed



x	y

More References

- Goldenberg, Paul, Philip Lewis, and James O'Keefe. "Dynamic Representation and the Development of a Process Understanding of Function." In The Concept of Function: *Aspects of Epistemology and Pedagogy*, edited by Ed Dubinsky and Guershon Harel, pp. 235–60. *MAA Notes* no. 25. Washington, D.C.: Mathematical Association of America, 1992.

More References

- http://www.geogebra.org/forum/viewtopic.php?f=2&t=22592&sd=d&start=15
- "Dynagraphs}--helping students visualize function dependency" • GeoGebra User Forum
- "degenerated" dynagraph game ("x" and "y" axes are superimposed) in GeoGebra: http://www.uff.br/cdme/c1d/c1d-html/c1d-en.html

More Think about These Problems

M.1 How would you use the Linear Focus to **find the mapping diagram for the function inverse for a linear function when $m \neq 0$?**

M.2 How does the **choice of axis scales** affect the **position of the linear function focus point** and its use in solving equations?

M.3 Describe the visual features of the mapping diagram for the quadratic function $f(x) = x^2$.
How does this generalize for *even* functions where

$$f(-x) = f(x)?$$

M.4 Describe the visual features of the mapping diagram for the cubic function $f(x) = x^3$.
How does this generalize for *odd* functions where

$$f(-x) = -f(x)?$$

More Think about These Problems

L.1 **Describe the visual features of the mapping diagram for the quadratic function** $\boldsymbol{f\,(x)\ =\ x^2}$.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.2 Describe the visual features of the mapping diagram for the quadratic function $\boldsymbol{f\,(x)\ =\ A(x-h)^2\ +\ k}$ using composition with simple linear functions.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.3 **Describe the visual features of a mapping diagram for the square root function** $\boldsymbol{g(x) = \sqrt{x}}$ **and relate them to those of the quadratic** $\boldsymbol{f\,(x)\ =\ x^2}$.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.4 **Describe the visual features of the mapping diagram for the reciprocal function** $\boldsymbol{f\,(x) = \frac{1}{x}}$.

Domain? Range? "Asymptotes" and "infinity"? Function Inverse?

L.5 Describe the visual features of the mapping diagram for the linear fractional function $\boldsymbol{f\,(x) = \frac{A}{x-h} +\ k}$ using composition with simple linear functions.
Domain? Range? "Asymptotes" and "infinity"? Function Inverse?

Thanks

The End!   REALLY!

☺

flashman@humboldt.edu

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