Two Different Approaches to Getting Students Involved in Writing Proofs .

Preliminary report.

10:45am to 10:55am. Friday, January 07, 2011

Martin E Flashman

Department of Mathematics

Humboldt State University

Arcata, CA 95521

flashman@humboldt.edu

Abstract

The author will present two different approaches he has used to engage students in developing and editing proofs- both important aspects of the art of writing.

- First is making a systematic analysis of a given proof to understand alternatives in style of presentation.
- The second is the use of "proofs without words" for exercises in transforming nonverbal thoughts and arguments into readable verbal presentations of the related argument or "proof."

References

Solow, Daniel. (1995). *The Keys to Advanced Mathematics: Recurrent Themes in Abstract Reasoning*. Cleveland Heights, Ohio: Books Unlimited.

Solow, Daniel. (2009). *How to Read and Do Proofs: An Introduction to Mathematical Thought Processes, 5th Edition*. New York, Wiley.

Learning to write (proofs?)

- The two exercise techniques
  - used in a sophomore/junior transition "introduction to proof" course.
  - address two distinct aspects of composition common to writing in many contexts:
  - **Analysis and deconstruction** of a composition: a skill used often in proof reading compositions.
  - **Articulation and transformation** of understanding: a skill used in initial creation and development of a composition.

SIGMAA RUME Session on Research on the Teaching and Learning of Undergraduate Mathematics, I - St. Jerome, 3rd Floor, JW Marriott

10:40 a.m. How mathematicians use diagrams to construct proofs.
11:20 a.m. Mathematics Majors' Evaluation of Mathematical Arguments and Their Conception of Proof.

From Abstracts to this meeting:1067-Z5-1339

Keith Weber* (keith.weber@gse.rutgers.edu), Graduate School of Education, 10 SeminaryPlace, New Brunswick, NJ 08840. Mathematics Majors' Evaluation of Mathematical Arguments and Their Conception of Proof.

Twenty-eight mathematics majors were observed as they read and evaluated ten mathematical arguments. The results of this study suggest that: (a) mathematics majors do not hold empirical proof schemes, as is widely believed, but many are convinced by perceptual arguments, (b) they will often accept an argument as \mostly correct" and as a valid proof even if they do not understand a pivotal claim within the argument, and (c) **many lack particular proof-reading skills needed to recognize the flaws in some arguments.**
(Received September 21, 2010)

Proof Analysis

- **Task: Analyze and deconstruct** a composition: proof reading.
- Make a **systematic structural and content** analysis of a given proof to understand the presentation.
  - Overview: What is the basic organization? Who is the intended audience?
  - Correctness: Is the argument sound? Definitions? Logic?
  - Coherence and Readability: Did the presentation of the argument make sense? Reader road signs? Omissions?
  - Alternatives: What might have improved the argument?

Proof Evaluation Example #1

Let $A=\{ x\in \mathbf{R} : x^2 + 1 < 10\}$; $B= \{ x\in \mathbf{R} : 0 < x < 2\}$

- **Proposition 1**: B is a subset of A. [$B \subseteq A$]
- **Proof**: By the definition of subset, we need to show that if r is an element of B then r is an element of A. Suppose r is an element of B. Then $0 < r < 2$, and thus

  $$0 < r^2 < 4 \text{ so that } 0 < r^2 +1 < 5.$$

  So it should be clear that r is an element of A. **EOP.**

Proposition 2: A is not a subset of B. [$A \not\subseteq B$]

- **Proof:** Consider the number -1, which is a member of A, but is not a member of B. Thus it is not the case that every member of A is a member of B. **EOP.**

Question 1

Let A={ $x\in$ **R** : $x^2 + 1 < 10$}; B= { $x\in$ **R** : $0 < x < 2$}

- **Proposition 1**: B is a subset of A. [$B \subseteq A$]
- **Proposition 2**: A is not a subset of B. [$A \not\subset B$]

1. Are the statements in propositions 1 and 2 conditional or absolute? [Note: For the purpose of this analysis a conditional statement is a statement that is expressed in the form; "If p then q", or "p implies q". An absolute statement is a statement that is not conditional.].

Question 2

Let $A = \{ x \in \mathbf{R} : x^2 + 1 < 10 \}$; $B = \{ x \in \mathbf{R} : 0 < x < 2 \}$

- **Proposition 1**: B is a subset of A. [$B \subseteq A$]
- **Proposition 2**: A is not a subset of B. [$A \not\subset B$]

2. List the variables used in these propositions. Indicate what these variables represent.

Question 3

Proof 1: By the definition of subset, we need to show that if r is an element of B then r is an element of A. Suppose r is an element of B. Then $0 < r < 2$, and thus

$$0 < r^2 < 4 \text{ so that } 0 < r^2 + 1 < 5.$$

So it should be clear that r is an element of A. **EOP.**

Proof 2: Consider the number -1, which is a member of A, but is not a member of B. Thus it is not the case that every member of A is a member of B. **EOP.**

3. Does the proof (for Proposition 1 and 2) proceed forward or is it mixed with some backward argument? If it has some backward argument, indicate briefly how the original conclusion is altered.

[“forward” “backward” see *How to Read and Do Proofs: An Introduction to Mathematical Thought Processes,* 5th Edition, Daniel Solow, ©2009.]

Question 4

Proof 1: By the definition of subset, we need to show that if r is an element of B then r is an element of A. Suppose r is an element of B. Then $0 < r < 2$, and thus

$$0 < r^2 < 4 \text{ so that } 0 < r^2 + 1 < 5.$$

So it should be clear that r is an element of A. **EOP.**

Proof 2: Consider the number -1, which is a member of A, but is not a member of B. Thus it is not the case that every member of A is a member of B. **EOP.**

4. Did the proofs <u>explicitly</u> leave some steps for the reader to complete?
If so, state what are steps the reader is expected to complete?
[Optional: complete these steps.]

Question 5

Proof 1: By the definition of subset, we need to show that if r is an element of B then r is an element of A. Suppose r is an element of B. Then $0 < r < 2$, and thus

$$0 < r^2 < 4 \text{ so that } 0 < r^2 + 1 < 5.$$

So it should be clear that r is an element of A. **EOP.**

Proof 2: Consider the number -1, which is a member of A, but is not a member of B. Thus it is not the case that every member of A is a member of B. **EOP.**

5. Did the proofs <u>implicitly</u> leave some steps for the reader to complete?
If so, state what steps you think the reader is expected to complete.

[Optional: Complete these steps.]

Question 6

Proof 1: By the definition of subset, we need to show that if r is an element of B then r is an element of A. Suppose r is an element of B. Then $0 < r < 2$, and thus

$$0 < r^2 < 4 \text{ so that } 0 < r^2 + 1 < 5.$$

So it should be clear that r is an element of A. **EOP.**

Proof 2: Consider the number -1, which is a member of A, but is not a member of B. Thus it is not the case that every member of A is a member of B. **EOP.**

6. Indicate any parts of the arguments that you felt needed greater detail or better connection to the proofs.

[Optional: Supply these details or suggest a better connection.]

Question 7

Proof 1: By the definition of subset, we need to show that if r is an element of B then r is an element of A. Suppose r is an element of B. Then $0 < r < 2$, and thus

$$0 < r^2 < 4 \text{ so that } 0 < r^2 + 1 < 5.$$

So it should be clear that r is an element of A. **EOP.**

Proof 2: Consider the number -1, which is a member of A, but is not a member of B. Thus it is not the case that every member of A is a member of B. **EOP.**

7. Overall, do you think these proofs were effective? Discuss briefly the basis for you conclusion.

Question 6

Proof 1: By the definition of subset, we need to show that if r is an element of B then r is an element of A. Suppose r is an element of B. Then $0 < r < 2$, and thus

$$0 < r^2 < 4 \text{ so that } 0 < r^2 + 1 < 5.$$

So it should be clear that r is an element of A. **EOP.**

Proof 2: Consider the number -1, which is a member of A, but is not a member of B. Thus it is not the case that every member of A is a member of B. **EOP.**

6. Indicate any parts of the arguments that you felt needed greater detail or better connection to the proofs.

[Optional: Supply these details or suggest a better connection.]

Proof Evaluation Example #2

- Reminder of the Definitions:
  (1) For a and b real numbers with a < b, **(a,b) ={ x : a < x < b}**
  (2) A set of real numbers, **O, is called an open set if and only if** for any number x that is a member of O there are some numbers a and b so that x is a member of (a,b) and (a,b)⊂O.
  (3) Suppose **I is a set and for each α in I, A_α is a set** .
  Then we define the intersection of the family A_α for α in I by
  ∩ A_α = {x : x is a member of A_α for every α in the set I} .

- **Proposition 1 : {5} is not an open set.**
  Proof: Suppose {5} is an open set.
  Consider the number 5, which is an element (in fact the only element) of {5}. Suppose a and b are any real numbers, where a < 5 < b. Then a < (5+a)/2 < 5 and therefore (a,b) is not a subset of {5}. Thus {5} is not an open set. EOP.

- **Proposition 2: [This proposition is FALSE.] If A_α is an open set of real numbers for every α in I, then ∩ A_α is an open set.**
  Proof: [This proof is erroneous.]
  Suppose x is a member of ∩ A_α . Then for every β in I, x is a member of A_β. Since A_β is an open set, there are real numbers a and b where x is a member of (a,b) and (a,b) is a subset of A_β for every β in I, and hence (a,b) is a subset of ∩ A_α . Therefore ∩ A_α is an open set. EOP.

Example #2 Question 1

- Reminder of the Definitions:
  (1) For a and b real numbers with $a < b$, **$(a,b) = \{ x : a < x < b\}$**
  (2) A set of real numbers, **O, is called an open set if and only if** for any number x that is a member of O there are some numbers a and b so that x is a member of (a,b) and $(a,b) \subset O$.
  (3) Suppose **I is a set and for each α in I, A_α is a set** .
  Then we define the intersection of the family A_α for α in I by
  $\cap A_\alpha = \{x : x$ is a member of A_α for every α in the set I$\}$.
- **Proposition 1 : {5} is not an open set.**
- **Proposition 2: [This proposition is FALSE.] If A_α is an open set of real numbers for every α in I, then $\cap A_\alpha$ is an open set.**
- **1. Are the statements in the propositions conditional or absolute?**
  If conditional, what are the hypotheses and conclusions?
  If absolute, can you rephrase the statement as a conditional statement?

Example #2 Question 2

Proof 1: Suppose {5} is an open set.
Consider the number 5, which is an element (in fact the only element) of {5}. Suppose a and b are any real numbers, where $a < 5 < b$. Then $a < (5+a)/2 < 5$ and therefore (a,b) is not a subset of {5}. Thus {5} is not an open set. EOP.

Proof 2: [This proof is erroneous.]
Suppose x is a member of $\cap A_\alpha$. Then for every β in I, x is a member of A_β. Since A_β is an open set, there are real numbers a and b where x is a member of (a,b) and (a,b) is a subset of A_β for every β in I, and hence (a,b) is a subset of $\cap A_\alpha$. Therefore $\cap A_\alpha$ is an open set. EOP.

2. Are the proofs of these propositions direct or indirect?

Example #2 Question 3

Proof 1: Suppose {5} is an open set.
Consider the number 5, which is an element (in fact the only element) of {5}. Suppose a and b are any real numbers, where $a < 5 < b$. Then $a < (5+a)/2 < 5$ and therefore (a,b) is not a subset of {5}. Thus {5} is not an open set. EOP.

Proof 2: [This proof is erroneous.]
Suppose x is a member of $\cap A_\alpha$. Then for every β in I, x is a member of A_β. Since A_β is an open set, there are real numbers a and b where x is a member of (a,b) and (a,b) is a subset of A_β for every β in I, and hence (a,b) is a subset of $\cap A_\alpha$. Therefore $\cap A_\alpha$ is an open set. EOP.

3. If the proof is indirect, state the way in which the argument proceeds.

(What is assumed? What is actually demonstrated?)

Example #2 Question 4

Proof 1: Suppose {5} is an open set.
Consider the number 5, which is an element (in fact the only element) of {5}. Suppose a and b are any real numbers, where $a < 5 < b$. Then $a < (5+a)/2 < 5$ and therefore (a,b) is not a subset of {5}. Thus {5} is not an open set. EOP.

Proof 2: [This proof is erroneous.]
Suppose x is a member of $\cap A_\alpha$. Then for every β in I, x is a member of A_β. Since A_β is an open set, there are real numbers a and b where x is a member of (a,b) and (a,b) is a subset of A_β for every β in I, and hence (a,b) is a subset of $\cap A_\alpha$. Therefore $\cap A_\alpha$ is an open set. EOP.

4. If the proof is direct, does the proof proceed forward or is it mixed with some backward argument? If it has some backward argument, indicate briefly how the original conclusion is altered.

Example #2 Question 5

- Reminder of the Definitions:
  (1) For a and b real numbers with a < b, **(a,b) ={ x : a < x < b}**
  (2) A set of real numbers, **O, is called an open set if and only if** for any number x that is a member of O there are some numbers a and b so that x is a member of (a,b) and $(a,b) \subset O$.
  (3) Suppose **I is a set and for each α in I, A_α is a set** .
  Then we define the intersection of the family A_α for α in I by
  $\cap A_\alpha$ = {x : x is a member of A_α for every α in the set I} .

Proposition 2: [This proposition is FALSE.] If A_α is an open set of real numbers for every α in I, then $\cap A_\alpha$ is an open set.

5. Proposition 2 is false.

Construct an example of a family of open sets so that the intersection of the family is {5}.
Why does your example show that proposition 2 is false?

Example #2 Question 6

Proof 2: [This proof is erroneous.]

Suppose x is a member of $\cap A_\alpha$. Then for every β in I, x is a member of A_β. Since A_β is an open set, there are real numbers a and b where x is a member of (a,b) and (a,b) is a subset of A_β for every β in I, and hence (a,b) is a subset of $\cap A_\alpha$. Therefore $\cap A_\alpha$ is an open set. EOP.

6. The proof of Proposition 2 has an error in it. Describe any and all errors you find in this "proof" of Proposition 2.

Example #2 Question 7

Proof 1: By the definition of subset, we need to show that if r is an element of B then r is an element of A. Suppose r is an element of B. Then $0 < r < 2$, and thus

$$0 < r^2 < 4 \text{ so that } 0 < r^2 + 1 < 5.$$

So it should be clear that r is an element of A. **EOP.**

7. Indicate any parts of the argument in Proposition 1 that you felt needed greater detail or better connection.

[Optional: Supply these details or suggest a better connection.]

Example #2 Question 8

Reminder of the Definitions:

(1) For a and b real numbers with a < b, **(a,b) ={ x : a < x < b}**

(2) A set of real numbers, **O, is called an open set if and only if** for any number x that is a member of O there are some numbers a and b so that x is a member of (a,b) and (a,b)$\subset$O.

(3) Suppose **I is a set and for each α in I, A_α is a set** .

Then we define the intersection of the family A_α for α in I by

$\cap A_\alpha$ = {x : x is a member of A_α for every α in the set I} .

Proposition 1 : {5} is not an open set.

Proof: Suppose {5} is an open set.

Consider the number 5, which is an element (in fact the only element) of {5}. Suppose a and b are any real numbers, where a < 5 < b. Then a < (5+a)/2 < 5 and therefore (a,b) is not a subset of {5}. Thus {5} is not an open set. EOP.

8. Generalize Proposition 1 and give a proof for your generalization.

Example #2 Question 9

Reminder of the Definitions:

(1) For a and b real numbers with a < b, **(a,b) ={ x : a < x < b}**

(2) A set of real numbers, **O, is called an open set if and only if** for any number x that is a member of O there are some numbers a and b so that x is a member of (a,b) and (a,b)⊂O.

(3) Suppose **I is a set and for each α in I, A_α is a set** .

Then we define the intersection of the family A_α for α in I by

∩ A_α = {x : x is a member of A_α for every α in the set I} .

Proposition 1 : {5} is not an open set.

Proof: Suppose {5} is an open set.

Consider the number 5, which is an element (in fact the only element) of {5}. Suppose a and b are any real numbers, where a < 5 < b. Then a < (5+a)/2 < 5 and therefore (a,b) is not a subset of {5}. Thus {5} is not an open set. EOP.

9. Overall, do you think the proof of proposition 1 was effective? Discuss briefly the basis for you conclusion.

Transforming: Non-verbal to Verbal

"proofs without words" exercises

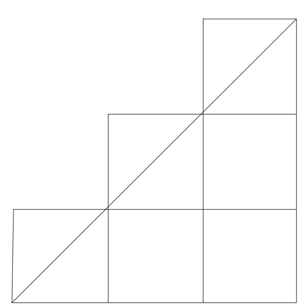
- **Task: Articulation and transformation** of understanding: a skill used in initial creation and development of a composition.
  - The exercise prompts the student to transform a figure that encompasses **nonverbal thoughts and arguments** (“proofs without words”) into a readable **verbal presentation of the related argument** or "proof."
  - Two examples to illustrate the prompt and the issues for the student in recognizing and transforming the argument.

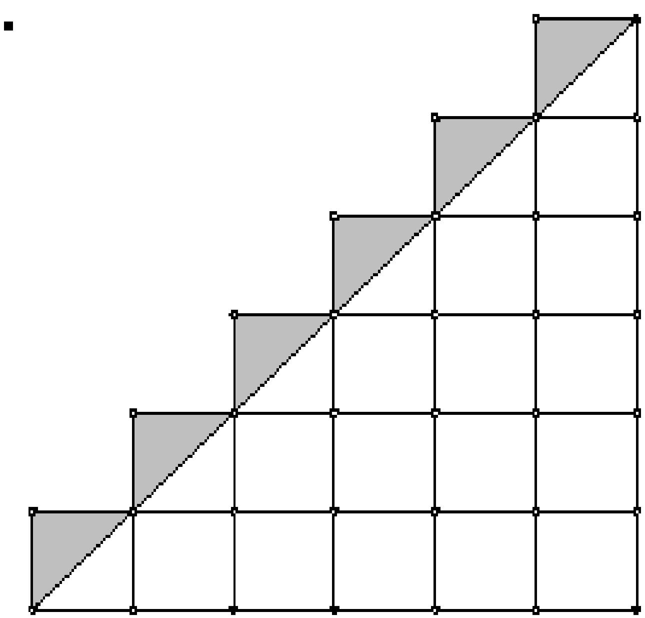
Proof without Words #1

The figures below contains information that can be used to explain why the following equation is true:

$$1 + 2 + . . . + n = n^2/2 + n/2.$$

Give a written explanation based on the figures for the truth of the equation.





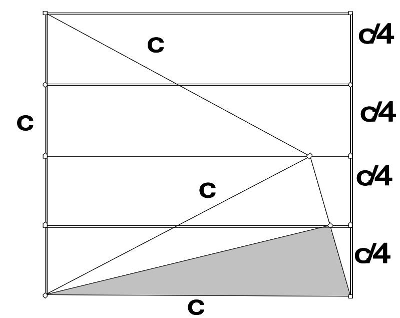
Issues in PWW #1

- Describe the figure geometrically to identify that the area is the sum, $1 + 2 + \ldots + n$.
- Recognize the one larger triangle ($b= n$, $h=n$) and the n smaller triangles ($b=1$, $h = 1$)
- Measure the areas of these triangles.
- Recognize the decomposition of the figure and identify the decomposition with the sum of the areas of triangles.

Proof without Words #2

The figure below contains information that can be used to explain why the following statement about the area of a right triangle is true:

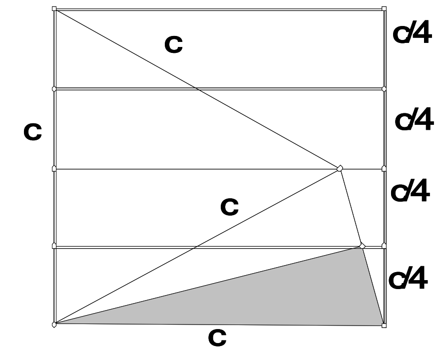
If one angle of a right triangle measures 15 degrees, then the area of the triangle is one eighth of the square of the hypotenuse.



Give a written explanation based on this figure by Klara Pinter for the truth of the statement. (Mathematics Magazine, V. 71, No.4, Oct., 1998)

BONUS:(POW) Prove the converse: If the area of a right triangle is one eighth of the square of the hypotenuse, then one angle of the triangle measures 15 degrees.

Issues with PWW #2



- Identify triangles in the square.
  - Equilateral triangle
  - 30 degree isosceles triangle
  - 15 degree right right
- Recognize the decomposition of the square into 4 congruent rectangles. Why are the sides all c/4?
- Compare the area of the 15 degree right triangle with the area of the square.

Conclusions?

- The two exercise techniques addressed two aspects of composition- learning to write:
  - **Analysis and decomposition** of a composition: a skill used in proof reading compositions.
  - **Articulation and transformation** of understanding: a skill used in initial creation and development of a composition.
- Further Research: Do these kinds of exercises improve student proof writing?

References[Again]

Solow, Daniel. (1995). *The Keys to Advanced Mathematics: Recurrent Themes in Abstract Reasoning*. Cleveland Heights, Ohio: Books Unlimited.

Solow, Daniel. (2009). *How to Read and Do Proofs: An Introduction to Mathematical Thought Processes, 5th Edition*. New York, Wiley.

Questions?

flashman@humboldt.edu

This presentation will be linked
on the web
by Jan.15 through

http://users.humboldt.edu/flashman

The End.

☺