Introduction to Sensible Calculus: A Thematic Approach

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Day by Day Outline (Rev'd 6-26)

0. Sunday: Basic Themes Plus …
   - Mapping Diagrams
   - Technology (Winplot and Geogebra)

I. Monday: Making Sense of the Derivative.

II. Tuesday: More on the Derivative

III. Wednesday: DE's, Approximation and The Fundamental Theorem of Calculus

IV. **Thursday: More on the FT, DE's, Models and Estimations**
   Making Sense of Taylor Theory

V. Friday: Frontiers-Probability, Economics, Series, …?

Continuing…

Newton's Method by Hand/Technology

- Using Spreadsheet: **Newton.xls**

- Using GeoGebra [New]: **Newton (ggb)**

Quick look at estimating important list of numbers:

$$\sqrt{2}, \pi, e, ln(2)$$

Euler's Method by Hand/Technology

- Using Spreadsheet: [Euler.xls]
- GeoGebra: [Euler (ggb)]
- Using Winplot: [diffeq.wp2]

Quick look at estimating important list of numbers:

$$\sqrt{2}, \pi, e, ln(2)$$

Euler's Method II
$y' = P(x, y)$

- Euler's method also works for solving an initial value problem:

Given $y' = P(x, y)$ **and** $f(a) = c$**, find exactly or estimate** $f(b)$.

  - One Step: the differential.
  - Two Equal Steps: the differential reset after first step using $y' = P(x, y)$.

– N Equal Steps: The differential reset after each step

  - Use of spread sheets to make the estimation systematic.

Euler's Method III
$y'' = P(x, y, y')$

- Euler's method also works for solving higher order initial value problems:

Given $y'' = P(x, y, y')$ **and** $f(a) = c, f'(a) = d$**, find exactly or estimate** $f(b)$.

  - One Step: the differential.
  - Two Equal Steps: the differential for y reset after first step using the differential for y'.

- N Equal Steps: The differential reset after each step

  - Use of spread sheets to make the estimation systematic.

Partner Problems
One Problem per Partner pair.

L.1 **Assume y is a solution to the differential equation**

$$\frac{dy}{dx} = \frac{1}{x^2+1} \text{ with } y(0) = 2.$$

(a) Using just the given information, find any local extreme points for y and discuss the graph of y, including the issue of concavity.
(b) Using the differential, estimate $y(1)$ and $y(-1)$.

L.2 Assume y is a solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{x^2+1}$$

(a) Sketch the tangent field showing tangents in all four quadrants.
(b) Draw three integral curves on your sketch **including one through the point $(\mathbf{1}, \mathbf{2})$;**
(c) Suppose that a solution to the differential equation has value 2 at 1.
 (i) Based on your graph, **estimate** the value of that solution at 2.
 (ii) Estimate the value of $y(3)$ using Euler's method with $n = 4$.

L.3 **Assume y is a solution to the differential equation**

$$\frac{dy}{dx} = -\frac{y}{x}$$

(a) Sketch the tangent field showing tangents in all four quadrants.
(b) Draw three integral curves on your sketch **including one through the point $(\mathbf{1}, \mathbf{2})$;**
(c) Suppose that a solution to the differential equation has value 2 at 1.
 (i) Based on your graph, **estimate** the value of that solution at 2.
 (ii) Estimate the value of $y(2)$ using Euler's method with $n = 4$.

L.4 Suppose $y'' = -y$, $y'(0) = 1$ and $y(0) = 0$. Estimate $y(1), y(2), y(3),$ and $y(4)$.

The Sensible Calculus Program

The Definite Integral, DE's, and Euler's Method

The motivation for defining the definite integral comes from estimating a solution to an Initial Value Problem, visual and numerical estimation with graphs and mapping diagrams.

V.A The Definite Integral – Connecting the definition to Euler's method and DE's.

The consequences of this approach-

The FT of C makes sense.

Sensible Calculus: Two Forms of the Fundamental Theorem of Calculus

Evaluation Form

If f is continuous and $G'(x) = f(x)$ for all x …. then

$$\int_{a}^{b} f(x)\,dx = G(b) - G(a).$$

Derivative Form (Barrow's Theorem)

If f is continuous and $G(t) = \int_a^t f(x)\,dx$ then

G is a differentiable function and $G'(t) = f(t)$.

Sensible Calculus: Two Forms of the Fundamental Theorem of Calculus

Derivative Form (Barrow's Theorem)

If f is continuous and $G(t) = \int_a^t f(x)\,dx$ then

G is a differentiable function and $G'(t) = f(t)$.

DE Interpretation: If f is continuous then there exists a solution to the DE: $G'(t) = f(t)$.

The Fundamental Theorem of Calculus
Derivative Form (Barrow's Theorem)

If $\boldsymbol{f}$ is continuous and $\boldsymbol{G(t) = \int_a^t f(x)\,dx}$ then $\boldsymbol{G}$ is a differentiable function and $\boldsymbol{G'(t) = f(t)}$.

Interpretation:

$f(x)$ is velocity of object at time x.

$G(t)$ is the net change in position of object from time a to time t.

$G'(t)$ = velocity of object at time t.

Sensible Calculus: Two Forms of the Fundamental Theorem of Calculus

Evaluation Form

If f is continuous and $G'(x) = f(x)$ for all x …. then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

DE Interpretation: If f is continuous then the initial value problem: $G'(t) = f(t)\ with\ G(a) = C$ has a unique solution:

$$\mathbf{G(t)} = \int_a^t f(x)\,dx + C.$$

Fundamental Theorem of Calculus
Evaluation Form

If f is continuous and $\boldsymbol{G'(x)} = \boldsymbol{f(x)}$ for all x ....
then $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(x)\,dx} = \boldsymbol{G(b)} - \boldsymbol{G(a)}$.

Interpretation:

$G(x)$ is a position function for a moving object which has its velocity at time x given by $f(x)$.

$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(x)\,dx}$ represents the net change in position of the object from time a to time b.

FT of Calculus
Objective & Key Ideas

Two Objectives:

- Estimate Net Change in Distance from differential equation using Euler's method for a derivative function that depends only on x
- Measure the error in using Euler's method to estimate net change for monotonic functions. [To ensure existence of integral for continuous functions]

Sensible Proofs of FT (Derivative)

- Background properties of definite integral make sense (with geometry or motion) Assume $\boldsymbol{a} < \boldsymbol{b}$.

$$\int_b^a f = -\int_a^b f$$

$$If\ \ m < \ f(x) < \ for\, all\, x \ldots then$$

$$m(b-a) < \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\,\boldsymbol{dx} < \boldsymbol{M}(\boldsymbol{b}-\boldsymbol{a})$$

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Sensible Proofs of FT (Derivative)

- Background properties of definite integral make sense (with geometry or motion) Assume $\boldsymbol{a} < \boldsymbol{b}$.

If $\boldsymbol{G(t)} = \int_{\boldsymbol{a}}^{\boldsymbol{t}} \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{dx}$ then use 4 steps to show $\boldsymbol{G}'(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{t})$.

Step I Interpretations: Geometry, Motion

Step II Interpretations: Geometry, Motion

Step III Interpretations: Geometry, Motion

Step IV THINK!

FT of Calculus
Objective & Key Ideas

Two Key Ideas:

- When x is close to a, f(x) is approximately equal to a linear function, $f(a) + f\,'(a)(x-a)$.
- As long as f is a sufficiently well behaved function there is some c between a and x where
  - $f(x) = f(a) + f\,'(c)(x-a)$.

Sensible Proofs of FT (Evaluation)

- Use Riemann Sum to estimate $\int_a^b f(x)\,dx$ then use Mean value theorem for each subinterval and telescoping sum!

  Interpret with motion and geometry.

- Use FT of C Derivative to justify FT of C Evaluation.

More on DE's, Models and Estimations

Pedagogy: Fundamental Concepts, not "foundations".

We continue by considering approaches that use models and connect with the familiar.

More on DE’s, Models and Estimations

Modeling contexts provide

- a sensible source for DE’s
- practical and theoretical use of concepts and skills.

Pedagogical decision: Engage students in balanced approaches to models that are

- visual,
- symbolic,
- numeric, and
- verbal.

More on DE's, Models and Estimations

- VI.A Differential Equations & Models- The Exponential Function
- VI.B Differential Equations & Models- The Natural Logarithm Function
- VI.C Connecting the Natural Logarithm and Exponential Functions
- VI.D More Models & Inverse Trigonometry
- IXA Taylor Theory for e^x

More on DE's, Models and Estimations

- VI.A Differential Equations & Models- The Exponential Function

Model Population:

- Biological cell growth
- Evolution of model from difference to differential equation with initial condition:
  - $P(0) = 1$
  - $P(t+1) - P(t) = P(t)$; $\Delta P = P$: Solution: $P(t) = 2^t$
  - $dP = Pdt$; $\frac{dP}{dt} = P$; Solution: ???
    - Direction Field
    - Euler's Method to estimate $P(1)$ and $P(t)$.
    - Exponential Properties evolve from the IVP. $P(t) = e^t$.
    - General solution for $\frac{dP}{dt} = k\,P$ with $P(0) = A$.

More on DE's, Models and Estimations

- VI.B Differential Equations & Models- The Natural Logarithm Function

Model Learning or slow growth Population:

- Evolution of model from heuristic on learning to differential equation with initial condition:
  - $L(1) = 0$
  - $L(t) > 0, L$ is increasing but at a decreasing rate due to fatigue. (Population growth but at a decreasing rate due to diminishing resources.)
  - $\frac{dL}{dt} = \frac{1}{t}$; Solution: ???
    - Direction Field
    - Euler's Method to estimate $L(1)$ and $L(t)$.
    - Connections to integration and FTof C
    - Logarithmic Properties evolve from the IVP. $L(t) = \ln(t)$.
    - General solution for $\frac{dL}{dt} = \frac{k}{\mathrm{t}}$ with $L(A) = 0$.
-

More on DE's, Models and Estimations

- VI.C Connecting the Natural Logarithm and Exponential Functions

More on DE's, Models and Estimations

- VI.D More Models & Inverse Trigonometry
- Other models for Learning and Poplulations
  - Bounded growth: $L'(t) = \frac{1}{t^2}$; $L(1) = 0$.
  - $L'(t) = \frac{1}{t^2+1}$; $L(0) = 0$.

Making Sense of Taylor Theory and the Calculus of Series

We continue by considering Taylor Theory through Modelling, Differential Equations, and Estimation.

What's Happening in The AP Calculus BC Course

- Applications: Series used for
  - **Estimation** of Numbers: $e, \pi, \sqrt{2}, \ldots$
  - **Estimation** of Definite Integrals.
  - Solution of **Differential Equations**.

What's Happening in
The AP Calculus BC Course

- **Critique**
  - **Little motivation from previous work**
    - **Newton's method.**
    - **Estimates of Definite Integrals.**
    - **Euler’s Solutions to Differential Equations.**
  - **Delayed Connection with the previous progress**
  - **Unclear statement of what is fundamental.**

Review: Sensible Calculus Approach

- **Estimating the solution to differential equations such as $y'' = -y$ or $y'' = y$ with $y(0) = 1$ and $y'(0) = 1$** provides motivation for convergence questions of infinite series.
  - Euler's method using $P(x, y, y') = -y \ or \ y$ also works for this!
  - $P(x, y, y') = -y$ can be connected to a model for simple harmonic motion (a spring).
- The calculus of Taylor polynomials (not series) is **a tool for approximating difficult definite integrals with a sensible control on the error.**

Review: Sensible Calculus Approach

- **From the beginning, the analysis of infinite** sequences and series **contains Taylor theory examples** along with traditional examples of geometric and harmonic series.
- **Historical connections:** Newton's work in estimating the values of $\ln(2)$ and π illustrates how geometric series played a significant role in showing the power of the early calculus for computation and estimation.

Start with Connections

- Review of previous work on estimates that have been sequential.
- Estimates for integrals:
  - constant,
  - Linear,
  - quadratic,
  - interpolation (trapezoid, simpson) versus single point based on derivative (midpoint).
- Symbolic and computational ease of polynomials for derivatives and integrals.

Illustrations from Sensible Calculus Taylor Theory and Series

Taylor Theory without series.

Motivating Example:

Section IX.A introducing Taylor Theory for e^x with applications.

http://users.humboldt.edu/flashman/book/ch9/IXA.htm

Taylor Theory-Objective & Key Ideas

Objectives:

- Find estimating polynomials for a given function

- Measure the error in using polynomials to estimate.

Taylor Theory-Objective & Key Ideas

Key Ideas:

- When x is close to a, $f(x)$ is approximately equal to a linear function, $\boldsymbol{f(a) + f'(a)(x-a)}$. **[Linear Approximation-Differential]**
- As long as f is a sufficiently well behaved function there is some c between a and x where $\boldsymbol{f(x) = f(a) + f'(c)(x-a)}$. **[MVT]**

Focus Themes for Series: Estimation, Differential Equations, Models

IX.A. Focus on **estimating a growth model with a differential equation:**

$$P'(x) = P(x), \qquad P(0) = 1.$$

- Solution is already treated
  - with estimation by Euler's method.
  - "Exactly":

$$P(x) = e^x$$

Focus Themes for Series: Estimation, Differential Equations, Models

IX.A. $\boldsymbol{P'(x) = P(x),\ P(0) = 1}$.

- Estimation of the solution: Use the polynomial of degree n that best matches the differential equation.
- Determine estimate of error for estimating
  - e
  - $\int_0^1 e^{-x^2}\, dx$

Discuss Sample Exercises from IX.A

1. Use the Taylor polynomial for e^x of degree 4 to estimate the following: $(a) e^2$ $(b) e^3$ $(c) e^{0.5}$ $(d) e^{-1}$ (e) $e^{3.14}$. [Spreadsheet helper supplied.]
2. Estimate e using the Taylor polynomial of degree n where n is (a) 6 (b) 7 (c) 8 (d) 10.
   In each estimate discuss the size of the error term R_n. [Spreadsheet helper.]
3. What value of n should be used so that the Taylor polynomial of degree n will give an estimate of e that is within .000001 of the exact value of e? Explain your result.
4. Use the Taylor polynomial for e^x of degree 5 to estimate $\int_0^1 e^{-t^2}\,dt$. Discuss the error in this approximation.

Sensible Calculus: EvolvingTaylorTheory

- IX.A Taylor Theory for e^x
- IX.B MacLaurin Polynomials and Taylor Theory
- IX.C MacLaurin Polynomials: How to Find Them
- IX.D Taylor Polynomials

From Polynomials to Power Series

- X.A Sequences
  - Examples
  - Definitions
  - Visualization
- X.B Series
  - Examples and Results:
    - Geometric Series
    - Harmonic Series
    - MacLaurin / Taylor Series
    - Binomial Series

From Polynomials to Power Series

- Tests for Divergence and Convergence.
  - Testing aimed at theme- power series and DE's from models.
  - Key examples:
    - Geometric Series
    - Harmonic Series
    - MacLaurin / Taylor Series
    - Binomial Series

Power Series: A Thematic Sensible Conclusion

- The Interval and Radius of Convergence.
- Functions and power series: $P(x) = \sum_0^\infty a_n x^n$.
  - Is P a differentiable function?
  - What is the derivative of P?
  - Is P a C^∞ function?
  - What are the Taylor polynomials and the Taylor series for P?
- Discussion: What is the relation of functions defined by power series to solving differential equations?

Conclusion for Thursday.

With this reorganization, the treatment of sequences and series forms **a sensible part of the first year calculus program**, capping an approach that focuses on understanding the three mathematical themes:

- **Differential Equations,**
- **Estimation, and**
- **Mathematical Modeling.**

End of Session IV

- Questions?

The End!

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Questions for next session?
Catch me between sessions or
e-mail them to me:
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