

Introduction to Sensible Calculus: A Thematic Approach



**The Anja S. Greer Conference on
Mathematics, Science and Technology
June 22 - June 27, 2014**

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Day by Day Outline (Rev'd 6-25)

0. Sunday: Basic Themes Plus ...

- Mapping Diagrams
- Technology (Winplot and Geogebra)

I. Monday: Making Sense of the Derivative.

II. Tuesday: More on the Derivative

III. Wednesday: DE's, Approximation and The Fundamental Theorem of Calculus

IV. Thursday: More on DE's, Models and Estimations. Making Sense of Taylor Theory and the Calculus of Series.

V. Friday: Frontiers-Probability, Economics, ...

Daily Assignment

Submit on paper or electronically.

- **Create one exercise and one problem** that incorporates (and/or extends) something from the session content.
- **Pose one question** related to the class content that you would like explained further. [I will respond privately unless you grant permission for a public response.]
- Take one (or two) topics discussed in the session and **discuss how you can incorporate** its content or technology into your teaching.
- **Electronic submissions may be shared with the class through the course webpage with submitter's permission.**
- **OPTIONAL:** Complete any worksheet or problems suggested during class.

Continuing from Last Class

Review GeoGebra Solution to Continuity Problem

- Download: [Cont Prob 2 \(ggb\)](#)

Making Sense of the Calculus of Derivatives

- Finding derivatives from the definition can be tedious for more complicated elementary functions.
- The calculus is a systematic procedure for finding the derivatives of elementary functions.
- An elementary function is a function built from a list of core functions by applying addition, subtraction, multiplication, division, and composition to the core functions and their inverses.
- The Core Functions (Short list): $c, x^n, e^x, \sin(x)$
- (Others) $x^r, b^x, \ln(x), \cos(x), \tan(x), \sec(x)$
- Rules: Linearity, Product, Quotient, Chain

Making Sense of a Differential Equation and the Fundamental Theorem of Calculus

- Example: The following differential equations of the form $\frac{dy}{dx} = P(x)$ have solutions that cannot be expressed as an elementary function.

$$- \frac{dy}{dx} = \sin(x^2)$$

$$- \frac{dy}{dx} = e^{-x^2}$$

- The solutions to these are given by using the FT of C:

$$y = f(t) = \int_0^t P(x) dx$$

The Fundamental Theorem of Calculus says:

When $P(x)$ is continuous, then $\frac{dy}{dt} = P(t)$.

The Fundamental Theorem of Calculus Derivative Form

If f is continuous and $G(t) = \int_a^t f(x) dx$ then
 G is a differentiable function and $G'(t) = f(t)$.

Interpretation:

$f(x)$ is velocity of object at time x .

$G(t)$ is the net change in position of object from time a to time t .

$G'(t) =$ velocity of object at time t .

Making Sense of Calculus: Applications to Estimation

- Intermediate Value Theorem, Roots and Continuity.
SC I.I.2. Intermediate Values
 - Bisection Algorithm
 - Graphical
 - Mapping Diagrams
 - Spreadsheets

Making Sense of Calculus: Applications to Estimation

- Linearity and Estimating Roots

III.A.2

- Linear Estimation Function:
 - Geometric Interpretation (Slope of Tangent line)
 - Motion Interpretation (Mapping Diagram, Magnification and Focus Point)
- Solving for roots in linear functions.
 - Brief excursion into inverses for linear functions.
 - More mapping diagrams!
- Newton's Method Algorithms. Estimation applications to error estimates.

Examples on Excel, Winplot, Geogebra

- **Excel example(s):**
 - [Linear Mapping Diagram example](#)
 - [Newtons Method](#)
- **Winplot examples:**
 - [Linear Mapping Diagram-composition examples](#)
 - [Linear Graph Linked File-composition examples](#)
- **Geogebra examples:**
 - [IV Steps](#)
 - [Secant Tangent](#)
 - [Alternative Derivative](#) for Sine.

Session III Differential Equations, Approximation and The Fundamental Theorem of Calculus

We continue to explore making calculus sensible by a consideration of the FT of Calculus from a view of DE's and estimations using Euler's Method interpreted in a variety of contexts.

Review: What's Happening Now in The First Calculus Course

- Differential Calculus: The derivative and applications- graphing, extremes, rates, Newton's method, mixing continuity and differentiability in theory, some slight mention of differential equations, THEN...
- Integral Calculus! Area, area, area, then Magic!
- The Fundamental Theorems of Calculus

What's Happening Now

Critique:

- Little motivation for integration from previous work despite
 - Local analysis of functions based on the derivative and MVT.
 - Estimation connected to
 - the derivative definition
 - linear approximating function (tangent line interpretation)
 - The differential
 - Introduction to Differential Equations available through implicit differentiation and related rates
- Unclear: What are the fundamental mathematical questions for a model?

Sensible Calculus: Make Connections

- Related Rates and Implicit differentiation involve “differential equations”
- Work on graphing using the derivative involves making qualitative inferences about a function from information about its derivative.
- Applications of the Mean Value Theorem suggest uniqueness of solution to IVP.
- Estimates using the differential (linear estimator).

Sensible Calculus: Two Forms of the Fundamental Theorem of Calculus

Evaluation Form

If f is continuous and $G'(x) = f(x)$ for all x ... then

$$\int_a^b f(x) dx = G(b) - G(a).$$

Derivative Form (Barrow's Theorem)

If f is continuous and $G(t) = \int_a^t f(x) dx$ then

G is a differentiable function and $G'(t) = f(t)$.

Fundamental Theorem of Calculus

Evaluation Form

If f is continuous and $G'(x) = f(x)$ for all x
then $\int_a^b f(x) dx = G(b) - G(a)$.

Interpretation:

$G(x)$ is a position function for a moving object which has its velocity at time x given by $f(x)$.

$\int_a^b f(x) dx$ represents the net change in position of the object from time a to time b .

The Fundamental Theorem of Calculus

Derivative Form (Barrow's Theorem)

If f is continuous and $G(t) = \int_a^t f(x) dx$ then G is a differentiable function and $G'(t) = f(t)$.

Interpretation:

$f(x)$ is velocity of object at time x .

$G(t)$ is the net change in position of object from time a to time t .

$G'(t) =$ velocity of object at time t .

The FT of Calculus, DE's, and Euler's Method

The motivation for the FT of C comes from estimating a solution to an Initial Value Problem: visual and numerical estimation with graphs and mapping diagrams.

Ch III.A.1. THE DIFFERENTIAL

Ch IV Differential Equations from an Elementary Viewpoint

V.A The Definite Integral - Connecting the definition to Euler's method and DE's.

Visualizing solutions to IVP's

Initial Value Problem (IVP) :

Given $y' = f'(x) = P(x)$ or $P(x, y)$ and $f(a) = c$, find exactly or estimate $f(b)$.

Connect the differential equation to the geometric interpretation using direction (tangent) fields and integral curves. Visual estimate of solution. See [Sensible Calculus](#) on DE's.

Estimating solutions to IVP's

Initial Value Problem (IVP) :

Given $y' = f'(x) = P(x)$ and $f(a) = c$,

find exactly or estimate $f(b)$.

Connect to previous work on estimates using the differential (linear estimator).

Euler's method evolves from a progression of estimates for solving an initial value problem.

Euler's Method

• Euler's method evolves from a progression of estimates for solving an initial value problem:

Given $y' = f'(x) = P(x)$ and $f(a) = c$, find exactly or estimate $f(b)$.

- One Step: the differential.
 - Two Equal Steps: the differential reset after first step.
 - N Equal Steps: The differential reset after each step
 - Use of spread sheets to make the estimation systematic.
- Ease of estimation of net change when $f'(x)$ depends only on x .

Euler's Method by Hand/Technology

- Using Spreadsheet and/or GeoGebra
 - [Euler.xls](#)
 - [Euler \(ggb\)](#)
- Using Winplot
 - [diffeq.wp2](#)

Euler's Method II

$$y' = P(x, y)$$

• Euler's method also works for solving an initial value problem:

Given $y' = P(x, y)$ and $f(a) = c$, find exactly or estimate $f(b)$.

- One Step: the differential.
- Two Equal Steps: the differential reset after first step using $y' = P(x, y)$.
- N Equal Steps: The differential reset after each step
 - Use of spread sheets to make the estimation systematic.

Euler's Method III

$$y'' = P(x, y, y')$$

• Euler's method also works for solving higher order initial value problems:

Given $y'' = P(x, y, y')$ and $f(a) = c, f'(a) = d$, find exactly or estimate $f(b)$.

- One Step: the differential.
- Two Equal Steps: the differential for y reset after first step using the differential for y' .
- N Equal Steps: The differential reset after each step
 - Use of spread sheets to make the estimation systematic.

Partner Problems

One Problem per Partner pair.

L.1 Assume y is a solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{x^2 + 1} \text{ with } y(0) = 2.$$

(a) Using just the given information, find any local extreme points for y and discuss the graph of y , including the issue of concavity.

(b) Using the differential, estimate $y(1)$ and $y(-1)$.

L.2 Assume y is a solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{x^2 + 1}$$

(a) Sketch the tangent field showing tangents in all four quadrants.

(b) Draw three integral curves on your sketch including one through the point $(1, 2)$;

(c) Suppose that a solution to the differential equation has value 2 at 1.

(i) Based on your graph, **estimate** the value of that solution at 2.

(ii) Estimate the value of $y(3)$ using Euler's method with $n = 4$.

L.3 Assume y is a solution to the differential equation

$$\frac{dy}{dx} = -\frac{y}{x}$$

(a) Sketch the tangent field showing tangents in all four quadrants.

(b) Draw three integral curves on your sketch including one through the point $(1, 2)$;

(c) Suppose that a solution to the differential equation has value 2 at 1.

(i) Based on your graph, **estimate** the value of that solution at 2.

(ii) Estimate the value of $y(2)$ using Euler's method with $n = 4$.

L.4 Suppose $y'' = -y$, $y'(0) = 1$ and $y(0) = 0$. Estimate $y(1)$, $y(2)$, $y(3)$, and $y(4)$.

The Sensible Calculus Program

The Definite Integral, DE's, and Euler's Method

The motivation for defining the definite integral comes from estimating a solution to an Initial Value Problem, visual and numerical estimation with graphs and mapping diagrams.

V.A The Definite Integral - Connecting the definition to Euler's method and DE's.

The consequences of this approach-

The FT of C makes sense.

FT of Calculus

Objective & Key Ideas

Two Objectives:

- Estimate Net Change in Distance from differential equation using Euler's method for a derivative function that depends only on x
- Measure the error in using Euler's method to estimate net change for monotonic functions.

FT of Calculus

Objective & Key Ideas

Two Key Ideas:

- When x is close to a , $f(x)$ is approximately equal to a linear function, $f(a) + f'(a)(x-a)$.
- As long as f is a sufficiently well behaved function there is some c between a and x where
$$f(x) = f(a) + f'(c)(x-a).$$

Conclusion

- With this reorganization, the treatment of the Fundamental Theorem of Calculus forms a sensible part of the first year calculus program, in a thematic approach to understanding the mathematical themes:

- Differential Equations,

- Estimation, and

- Mathematical Modeling

Examples on Excel, Winplot, Geogebra

- **Excel example(s):**
 - Euler's Method
- **Winplot examples:**
 - Linear Mapping Diagram-composition examples
 - Linear Graph Linked File-composition examples
- **Geogebra examples:**
 - Euler's Method with Mapping diagram & rectangles

End of Session III



**Questions for next session?
Catch me between sessions or
e-mail them to me:**

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- [FL1] Flashman, Martin. "[Differential Equations: A Motivating Theme for A Sensible Calculus](#)," in "Calculus for All Users" The Report of A Conference on Calculus and Its Applications Held at the University of Texas, San Antonio, NSF Calculus Reform Conference, October 5 - 8, 1990.
- [UMAP] Flashman, Martin. "[A Sensible Calculus](#)," The UMAP Journal, Vol. 11, No. 2, Summer, 1990, pp. 93-96.
- [FL2] Flashman, Martin. "Using Computers to Make Integration More Visual with Tangent Fields," appearing in Proceedings of the Second Annual Conference on Technology in Collegiate Mathematics, Teaching and Learning with Technology of November 2-4, 1989, edited by Demana, Waits, and Harvey, Addison-Wesley, 1991.
- [FL3] Flashman, Martin. "Concepts to Drive Technology," in Proceedings of the Fifth Annual Conference on Technology in Collegiate Mathematics, November 12-15, 1992, edited by Lewis Lum, Addison-Wesley, 1994.
- [FL4] Flashman, Martin. "Historical Motivation for a Calculus Course: Barrow's Theorem," in Vita Mathematica: Historical Research and Integration with Teaching, edited by Ronald Calinger, MAA Notes, No. 40, 1996.