

Introduction to Sensible Calculus: A Thematic Approach



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Day by Day Outline

0. Sunday: Basic Themes Plus ...

- Mapping Diagrams
- Technology (Winplot and Geogebra)

I. Monday: Making Sense of the Derivative.

II. Tuesday: DE's, Approximation and The Fundamental Theorem of Calculus

III. Wednesday: More on DE's, Models and Estimations

IV. Thursday: Making Sense of Taylor Theory and the Calculus of Series

V. Friday: Frontiers-Probability, Economics, ...

Daily Assignment

Submit on paper or electronically.

- **Create one exercise and one problem** that incorporates (and/or extends) something from the session content.
- **Pose one question** related to the class content that you would like explained further. [I will respond privately unless you grant permission for a public response.]
- Take one (or two) topics discussed in the session and **discuss how you can incorporate** its content or technology into your teaching.
- **Electronic submissions may be shared with the class through the course webpage with submitter's permission.**
- **OPTIONAL:** Complete any worksheet or problems suggested during class.

Continuing from Last Class

Sensible Calculus and Mapping Diagram Resources

- [Sensible Calculus Visualizations and Mapping Diagrams](#)
- [Mapping Diagrams from A\(lgebra\) B\(asics\) to C\(alculus\) and D\(ifferential\) E\(quation\)s.](#)

A Reference and Resource Book on
Function Visualizations Using Mapping
Diagrams

Examples with Technology

[LINK for Current Materials](#)

- Excel examples [Covered last class]
- Winplot examples
- Geogebra examples
- SketchPad examples

Simple Examples are important!

$f(x) = mx + b$ with a mapping diagram -

Five examples:

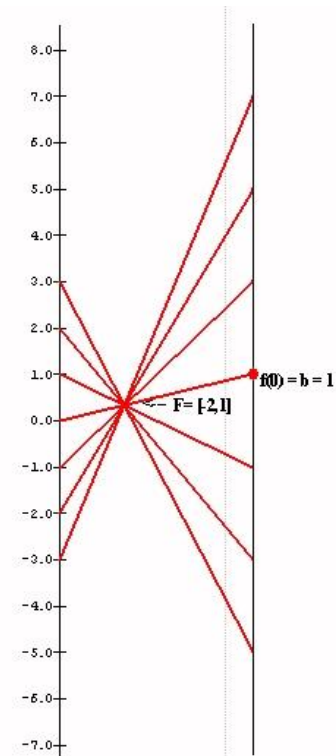
- Example 1: $m = -2$; $b = 1$: $f(x) = -2x + 1$
- Example 2: $m = 2$; $b = 1$: $f(x) = 2x + 1$
- Example 3: $m = \frac{1}{2}$; $b = 1$: $f(x) = \frac{1}{2}x + 1$
- Example 4: $m = 0$; $b = 1$: $f(x) = 0x + 1$
- Example 5: $m = 1$; $b = 1$: $f(x) = x + 1$

Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples

Example 1: $m = -2$; $b = 1$

$$f(x) = -2x + 1$$

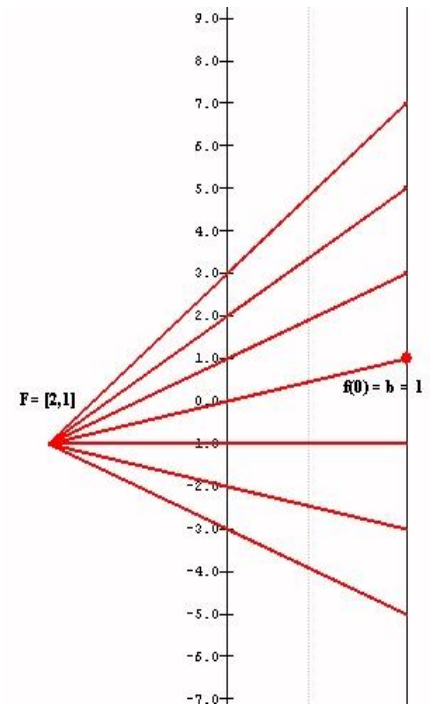
- Each arrow passes through a single point, which is labeled $F = [-2, 1]$.
 - The point F completely determines the function f .
 - given a point / number, x , on the source line,
 - there is a unique arrow passing through F
 - meeting the target line at a unique point / number, $-2x + 1$,
- which corresponds to the linear function's value for the point/number, x .



Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples:

Example 2: $m = 2; b = 1$
 $f(x) = 2x + 1$

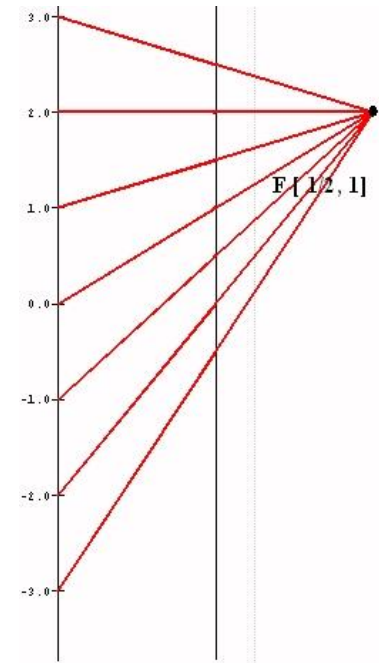
- Each arrow passes through a single point, which is labeled $F = [2, 1]$.
- The point F completely determines the function f .
 - given a point / number, x , on the source line,
 - there is a unique arrow passing through F
 - meeting the target line at a unique point / number, $2x + 1$,which corresponds to the linear function's value for the point/number, x .



Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

Example 3: $m = \frac{1}{2}; b = 1$
 $f(x) = \frac{1}{2}x + 1$

- Each arrow passes through a single point, which is labeled $F = [\frac{1}{2}, 1]$.
- The point F completely determines the function f .
 - given a point / number, x , on the source line,
 - there is a unique arrow passing through F
 - meeting the target line at a unique point / number, $\frac{1}{2}x + 1$,which corresponds to the linear function's value for the point/number, x .

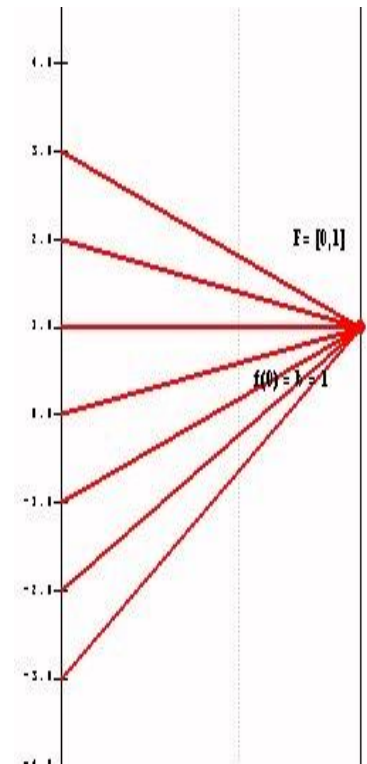


Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

Example 4: $m = 0; b = 1$

$$f(x) = 0x + 1$$

- Each arrow passes through a single point, which is labeled $F = [0, 1]$.
 - The point F completely determines the function f .
 - given a point / number, x , on the source line,
 - there is a unique arrow passing through F
 - meeting the target line at a unique point / number, $0x + 1$,
- which corresponds to the linear function's value for the point/number, x .



Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

Example 5: $m = 1; b = 1$

$$f(x) = 1x + 1$$

- Unlike the previous examples, in this case it is not a single point that determines the mapping figure, but the single arrow from 0 to 1, which we designate as $F[1,1]$
- It can also be shown that this single arrow completely determines the function. Thus, given a point / number, x , on the source line, there is a unique arrow passing through x **parallel** to $F[1,1]$ meeting the target line at a unique point / number, $x + 1$, which corresponds to the linear function's value for the point/number, x .
- The single arrow completely determines the function f .
 - given a point / number, x , on the source line,
 - there is a **unique arrow** through x **parallel** to $F[1,1]$
 - meeting the target line at a **unique point** / number, $x + 1$, which corresponds to the linear function's value for the point/number, x .



I Making Sense of Calculus: The Derivative.

Motivation from Models

Balance in Interpretations

Exact vs. Estimated

Local Linearity

Unification/Generalization

Abstraction/Application (D. Solow)

Session I Making Sense of the Derivative.

Reconsideration of the linear function

$$y = f(x) = mx + b$$

Interpretation Model context	x	$y = f(x)$	m
Motion	(t) time	(s) position on a coordinate line	Velocity of object (Constant)
Coordinate Geometry	1 st coordinate of a point in the plane	2nd coordinate of a point in the plane	Slope of line
Economic Production	quantity of a product bought/sold	total cost/revenue/profit for product bought/sold	marginal cost/revenue/profit of product (Constant)
Probability Random Variable	range value of a given random variable	probability that the random variable will be less than or equal to the given range	probability density of random variable

Making Sense of Calculus: The Derivative.

- Ch 0 A Motivation: What is the calculus?
- Ch 0 B Solving the Tangent Problem

The derivative as a number:

- a modelling context
- visual and numerical estimation
 - graphs
 - mapping diagrams.
- Ch 1 A Tangent Line
- Ch 1 B Velocity

Making Sense of Calculus: The Derivative.

- Ch 0 A Motivation: What is the calculus?

Making Sense of Calculus: The Derivative.

- Ch 0 A Motivation: What is the calculus?

The derivative as a number:

- a modelling context
- visual and numerical estimation
 - graphs (Slope)
 - mapping diagrams (Magnification)

Making Sense of Calculus: The Derivative.

- Ch 0 B Solving the Tangent Problem

The derivative as a number:

- a modelling context
- visual and numerical estimation
 - graphs (Slope)

Making Sense of Calculus: The Derivative.

The derivative as a number:

- a modelling context
- visual and numerical estimation
 - graphs (Slope)
- Ch 1 A Tangent Line
 - Estimation
 - Four Steps
- **UNIFICATION /GENERALIZATION**

Making Sense of Calculus: The Derivative.

The derivative as a number:

- a modelling context
- visual and numerical estimation
 - mapping diagrams (Magnification)
- Ch 1 B Velocity
 - Estimation
 - Four Steps
- **UNIFICATION /GENERALIZATION**

Making Sense of Calculus: The Derivative.

The derivative as a number:

- **ABSTRACTION**
- visual and numerical estimation
 - graphs (Slope)
 - mapping diagrams (Magnification)
- Ch 1 D Derivative
 - Estimation
 - Four Steps

Making Sense of Calculus: The Derivative.

The derivative as a number:

- a modelling context:
Generalization/Abstraction/Application
- visual and numerical estimation
 - graphs (Slope)
 - mapping diagrams (Magnification)
- Ch 1 C Other Models/Interpretations for the Derivative
 - Probability: (Point) Probability Density
 - Economics: Marginal Cost/Revenue/Profits

Think about These Problems

M.1 Use a mapping figure for the function $f(x) = -3x + 2$ to illustrate that

$$f'(1) = -3.$$

Sketch a mapping figure that illustrates the work to show that the linear function $f(x) = mx + b$ has $f'(a) = m$. Discuss how different values of m impact your figure.

M.2 Use a mapping figure for the function $f(x) = x^2$ to illustrate that

$$f'(3) = 6.$$

Sketch a mapping figure that illustrates the work to show that $f'(a) = 2a$.

M.3 Use a mapping figure for the function $f(x) = \frac{1}{x}$ to illustrate that

$$f'(2) = -\frac{1}{4}.$$

Sketch a mapping figure that illustrates the work to show that $f'(a) = -\frac{1}{a^2}$.

Making Sense of Calculus: The Derivative of Core Functions

- The Exponential Function

SC I.F.2

- Motivate with Population Model:

$$P(t) = 2^t$$

- Find $P'(0)$.
 - Connect $P'(t) = P'(0) P(t)$
- Connect to Compound Interest Rate
- Interpretation and definition of e .
- If $P'(x) = P(x)$ with $P(0) = 1$ then $P(x) = e^x$.

Making Sense of Calculus: The Derivative of Core Functions

- *The Sine Function*

I.F.3

- *Motivate with Unit Circle Motion:*

$$f(t) = \sin(t), g(t) = \cos(t)$$

- *Estimate, then find $f'(0), g'(0)$.*
- *Connect $f'(t) = f'(t)g(0) + g'(t)f(0)$*

Making Sense of Calculus: The Derivative Calculus

- Product Rule SC [II.A](#)
- Motivate with Linearity in Algebra
 - Linear Estimation
- Connect to Rate Interpretation
 - Rectangular Area
 - Mapping Diagram of Sides
 - Error Estimation
- Continuity and Differentiability Connection

Making Sense of Calculus: The Derivative Calculus

- Chain Rule SC [II.B](#)
- Motivate with Linearity in Algebra
 - Linear Estimation
- Connect to Rate Interpretation
 - Gas consumption, Motion, Time
 - Mapping Diagram for Composition
 - Error Estimation
 - Pattern Recognition
 - Leibnitz Notation

Making Sense of Calculus: Applications to Estimation

- Local Linearity and the Differential

III.A.1

- Linear Estimation Function:
 - Geometric Interpretation (Slope of Tangent line)
 - Motion Interpretation (Mapping Diagram, Magnification and Focus Point)
- Leibniz Notation and the Differential
- Estimation applications to error estimates.

Making Sense of Calculus: Applications to Estimation

- Intermediate Value Theorem, Roots and Continuity.

SC I.I.2. Intermediate Values

- Bisection Algorithm
 - Graphical
 - Mapping Diagrams
- Spreadsheets

Making Sense of Calculus: Applications to Estimation

- Linearity and Estimating Roots

III.A.2

- Linear Estimation Function:
 - Geometric Interpretation (Slope of Tangent line)
 - Motion Interpretation (Mapping Diagram, Magnification and Focus Point)
- Solving for roots in linear functions.
 - Brief excursion into inverses for linear functions.
 - More mapping diagrams!
- Newton's Method Algorithms Estimation

Examples on Excel, Winplot, Geogebra

- **Excel example(s):**
 - [Linear Mapping Diagram example](#)
 - [Newtons Method](#)
- **Winplot examples:**
 - [Linear Mapping Diagram-composition examples](#)
 - [Linear Graph Linked File-composition examples](#)
- **Geogebra examples:**
 - [IV Steps](#)
 - [Secant Tangent](#)
 - [Alternative Derivative](#) for Sine.

End of Session I

- Next session will deal further with inferential applications of the derivative as a transition to the study of differential equations, integration and the Fundamental Theorem of Calculus
- Questions?

**Thanks
The End!**



**Questions for next session?
Catch me between sessions or
e-mail them to me:**

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- [FL1] Flashman, Martin. "[Differential Equations: A Motivating Theme for A Sensible Calculus](#)," in "Calculus for All Users" The Report of A Conference on Calculus and Its Applications Held at the University of Texas, San Antonio, NSF Calculus Reform Conference, October 5 - 8, 1990.
- [UMAP] Flashman, Martin. "[A Sensible Calculus](#)," The UMAP Journal, Vol. 11, No. 2, Summer, 1990, pp. 93-96.
- [FL2] Flashman, Martin. "Using Computers to Make Integration More Visual with Tangent Fields," appearing in Proceedings of the Second Annual Conference on Technology in Collegiate Mathematics, Teaching and Learning with Technology of November 2-4, 1989, edited by Demana, Waits, and Harvey, Addison-Wesley, 1991.
- [FL3] Flashman, Martin. "Concepts to Drive Technology," in Proceedings of the Fifth Annual Conference on Technology in Collegiate Mathematics, November 12-15, 1992, edited by Lewis Lum, Addison-Wesley, 1994.
- [FL4] Flashman, Martin. "Historical Motivation for a Calculus Course: Barrow's Theorem," in Vita Mathematica: Historical Research and Integration with Teaching, edited by Ronald Calinger, MAA Notes, No. 40, 1996.
- [DS1] Solow, Daniel. [The Keys to Advanced Mathematics : Recurrent Themes in Abstract Reasoning](#)
- [DS2] Solow, Daniel. **How to Read and Do Proofs: An Introduction to Mathematical Thought Processes**, Wiley; 6 ed (July 29, 2013).