Binomial Identities with $PascalGT$

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PascalGT Developers and References

M. Bardzell, K. Shannon and T. Wilson (Salisbury University)

M. J. Bardzell and K. M. Shannon, *The PascGalois Triangle: A Tool for Visualizing Abstract Algebra*, Innovations in Teaching Mathematics, MAA Notes, **60** (2002), 115-123.

M. J. Bardzell and K. M. Shannon, PascGalois web site, `http://faculty.salisbury.edu/~kmshannon/pascal/`

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Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & & & & & & \vdots & & & & & & \\
1 & & \binom{n}{1} & & \binom{n}{2} & & \cdots & & \binom{n}{n-2} & & \binom{n}{n-1} & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

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\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & & & & & & \vdots & & & & & & \\
1 & \binom{n}{1} & & & \binom{n}{2} & & \cdots & & \binom{n}{n-2} & & \binom{n}{n-1} & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}, \; n \geq 2, \; 1 \leq k \leq n-1$$

Pascal's Triangle modulo m

Reduce all entries in Pascal's triangle modulo m for a fixed integer $m \geq 1$.

Equivalently, use addition modulo m to generate the interior.

Addition modulo m is the group law in the cyclic group $\mathbb{Z}_m$.

The above construction generalizes immediately to abstract groups.

PascGalois Triangles

If G is any group, a and b are two (possibly equal) elements of G, we place $a \in G$ down the left side of the triangle and $b \in G$ down the right hand side, we can use the group multiplication in G to fill in the interior of the triangle.

The resulting triangle is called a **PascGalois triangle**, and is denoted by $P_G(a, b)$.

PascGalois Triangles

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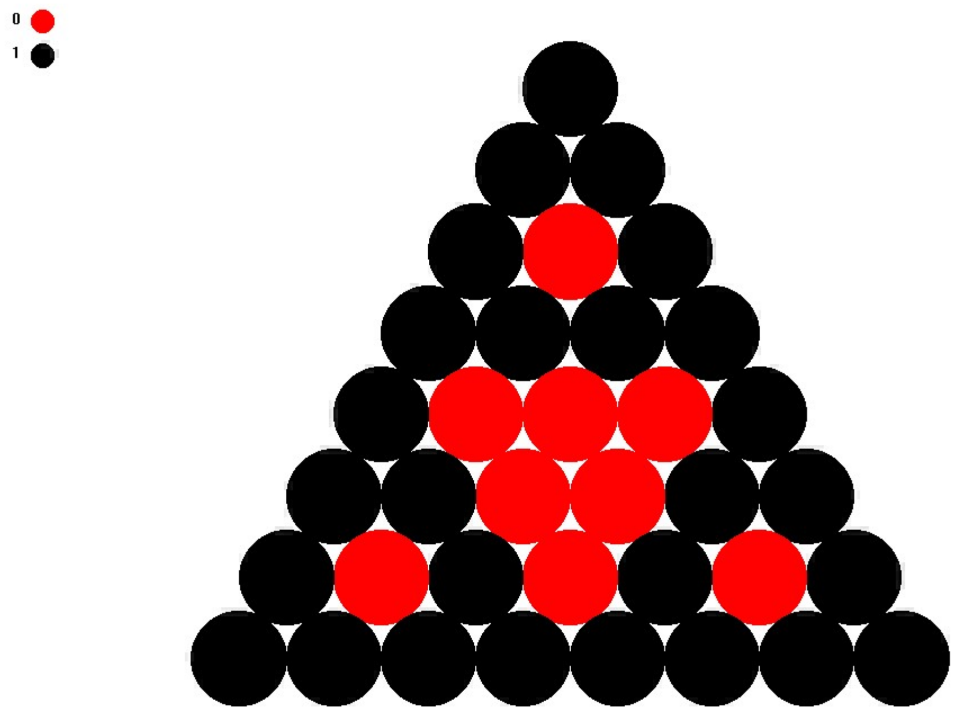
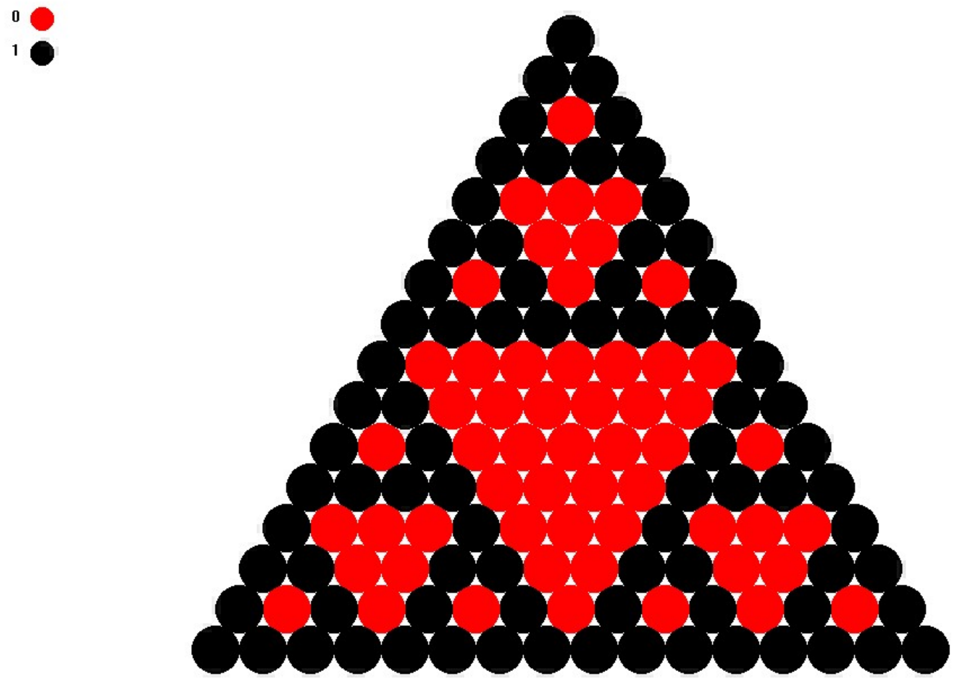
$P_{\mathbb{Z}}(1, 1)$is Pascal's triangle. $P_{\mathbb{Z}_m}(1, 1)$ is Pascal's triangle modulo m.

$PascalGT$ assigns a unique color to each element of a finite group G, and generates images of PascGalois triangles with a specified number of rows.

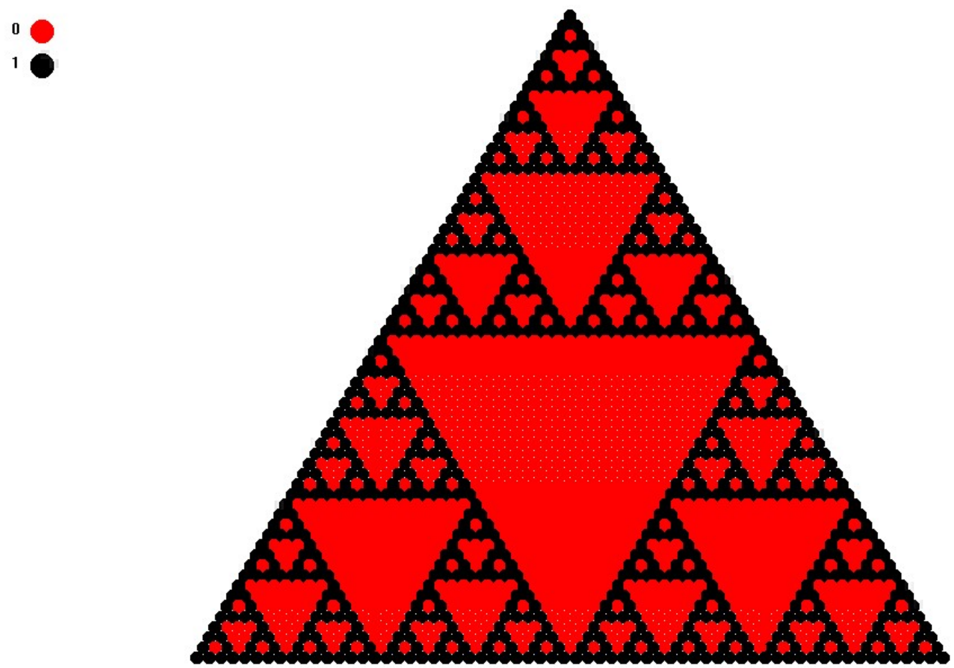
$P_{\mathbb{Z}_2}(1,1)$ with $N = 7$



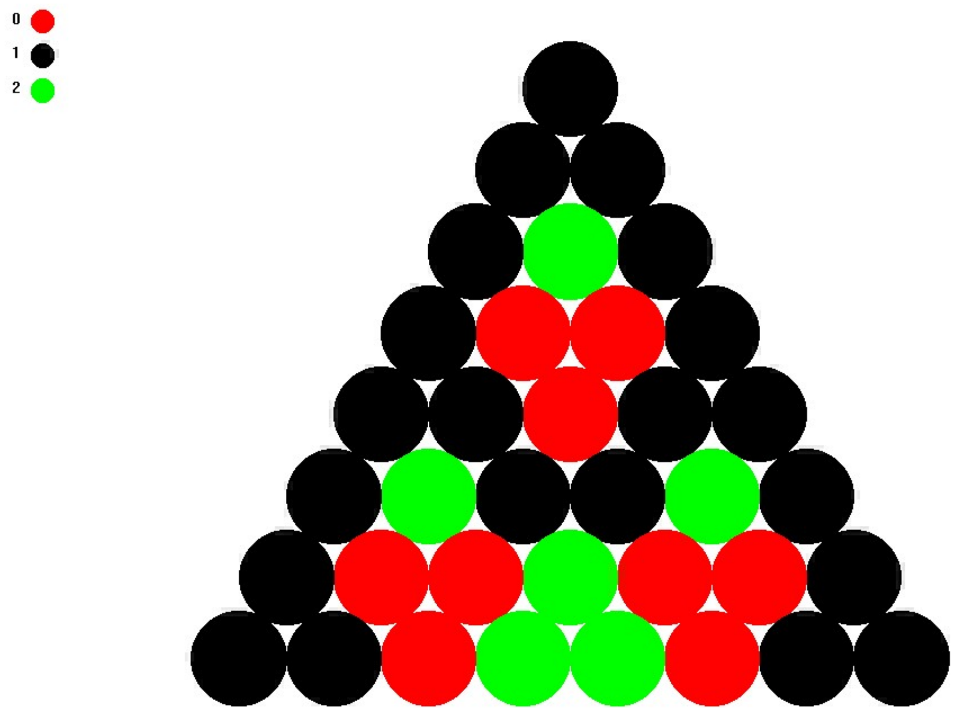
$P_{\mathbb{Z}_2}(1,1)$ **with** $N = 15$



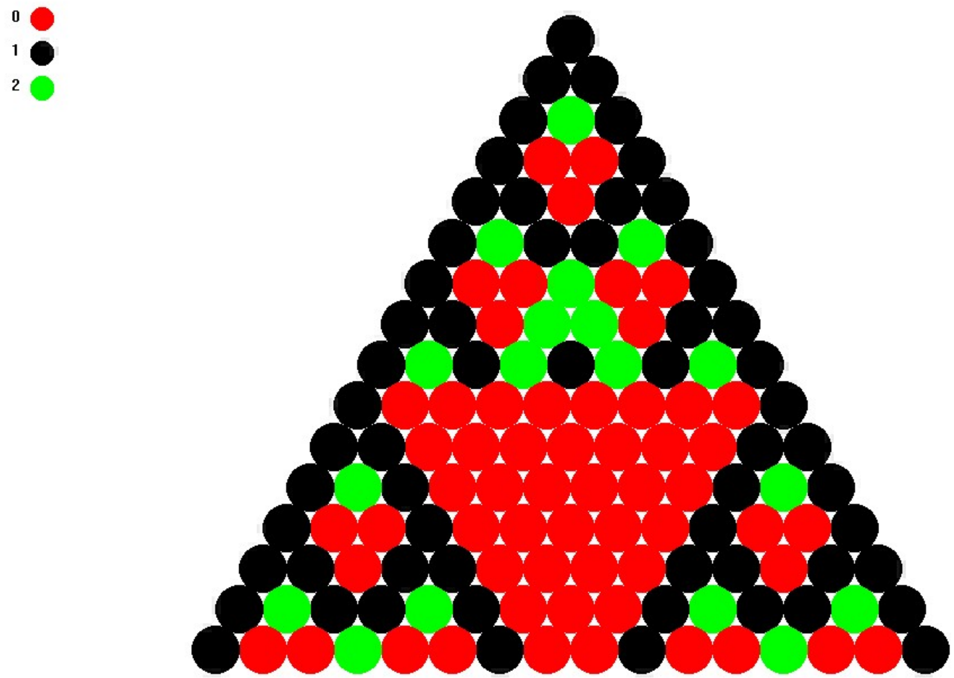
$P_{\mathbb{Z}_2}(1,1)$ **with** $N = 63$



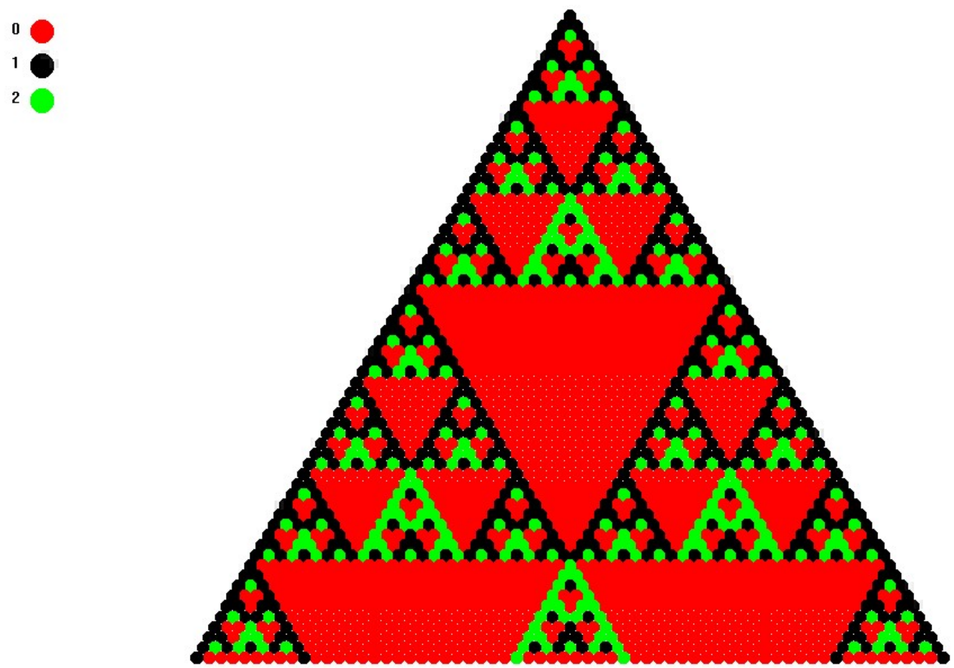
$P_{\mathbb{Z}_3}(1,1)$ with $N = 7$



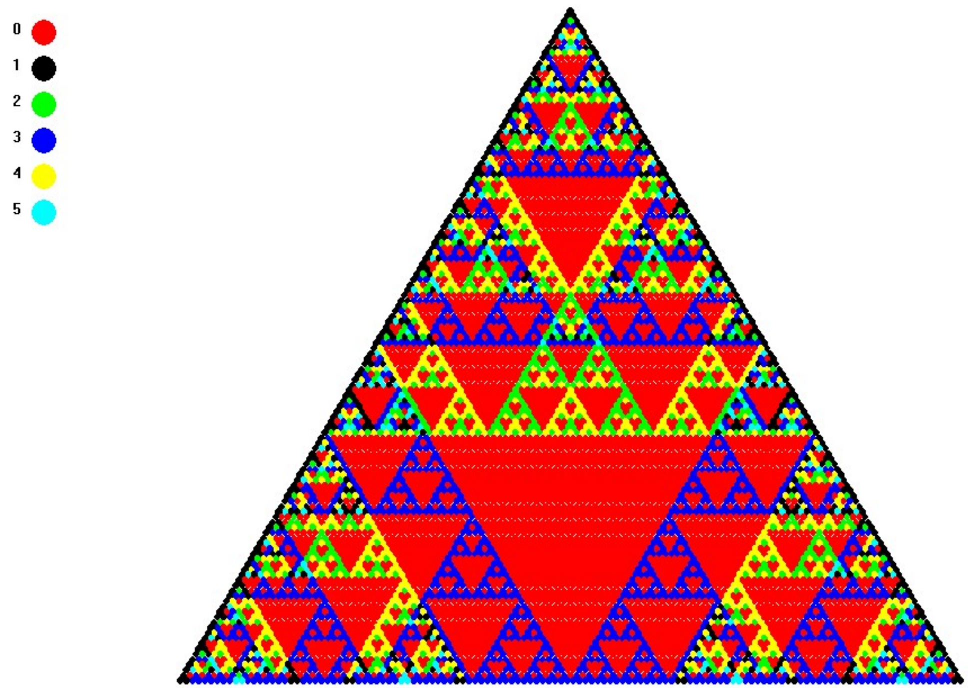
$P_{\mathbb{Z}_3}(1,1)$ **with** $N = 15$



$P_{\mathbb{Z}_3}(1,1)$ with $N = 63$



$P_{\mathbb{Z}_6}(1,1)$ **with** $N = 127$



Exercise: Freshman Exponentiation

For $n \geq 0$ and $0 \leq k \leq n$, let

$$\left[\begin{array}{c} n \\ k \end{array}\right] = \binom{n}{k} \pmod{m}.$$

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The usual binomial theorem holds in the ring $\mathbb{Z}_m$:

$$(a+b)^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} a^{n-j} b^j, a, b \in \mathbb{Z}_m.$$

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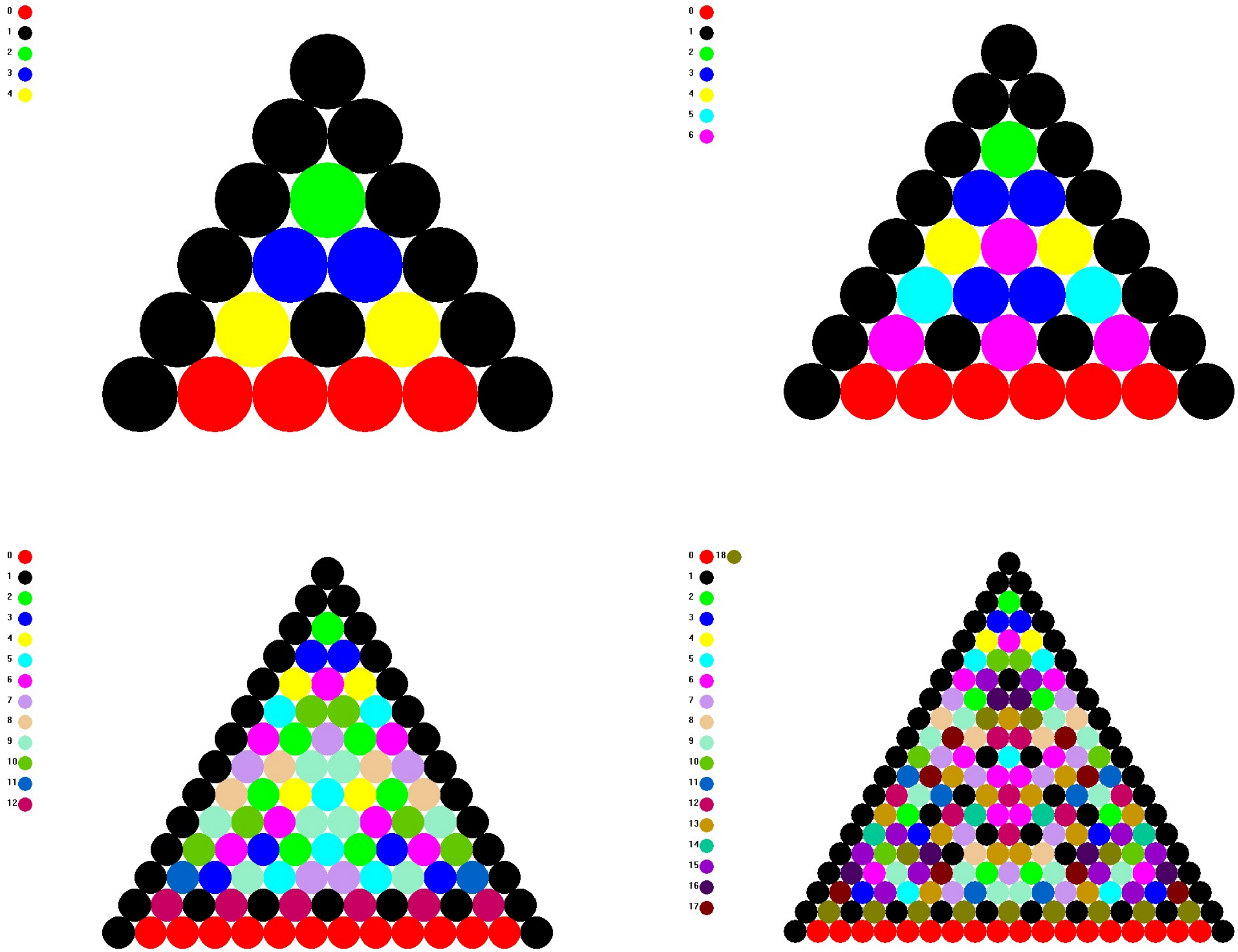
Generate $P_{\mathbb{Z}_p}(1,1)$ for various primes p. Look at the entries in the row $n = p$ and make a conjecture about $(a+b)^p$.

Conjecture: $(a+b)^p = a^p + b^p$



Pascal's Rule Rearranged

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Pascal's Rule Rearranged

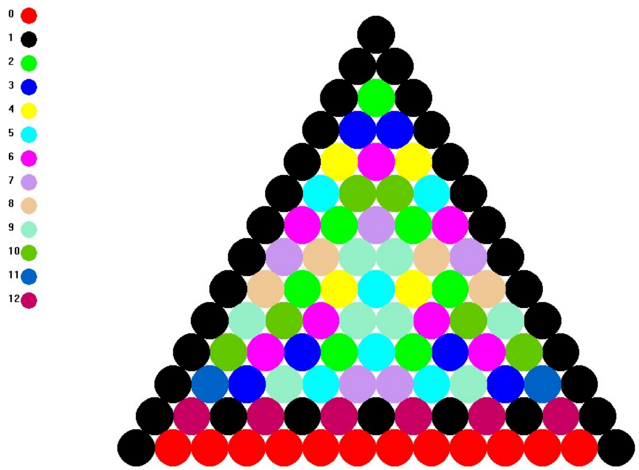
$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

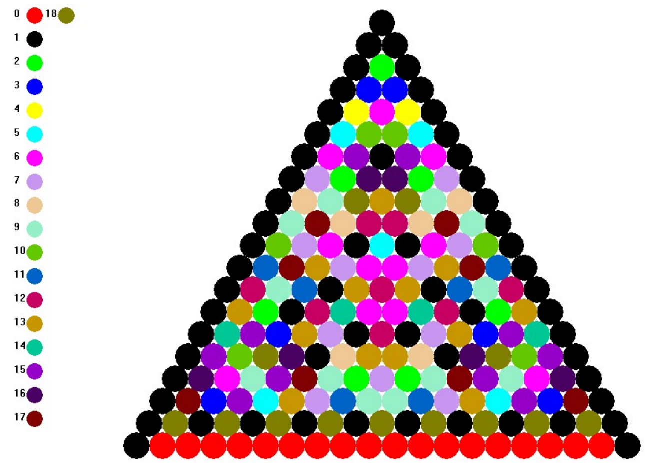
For $n \geq 2$ and $1 \leq k \leq n-1$:

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k} + \binom{n-2}{k-1} + \cdots + \binom{n-1-(k-1)}{1} + \binom{n-1-k}{0} \\ &= \sum_{j=0}^{k} \binom{n-1-k+j}{j} \end{aligned}$$

Formulate a conjecture for $\displaystyle\sum_{j=0}^{k} \binom{p-1-k+j}{j}$, $1 \leq k \leq p-1$ for primes p.

Pascal's Rule Rearranged



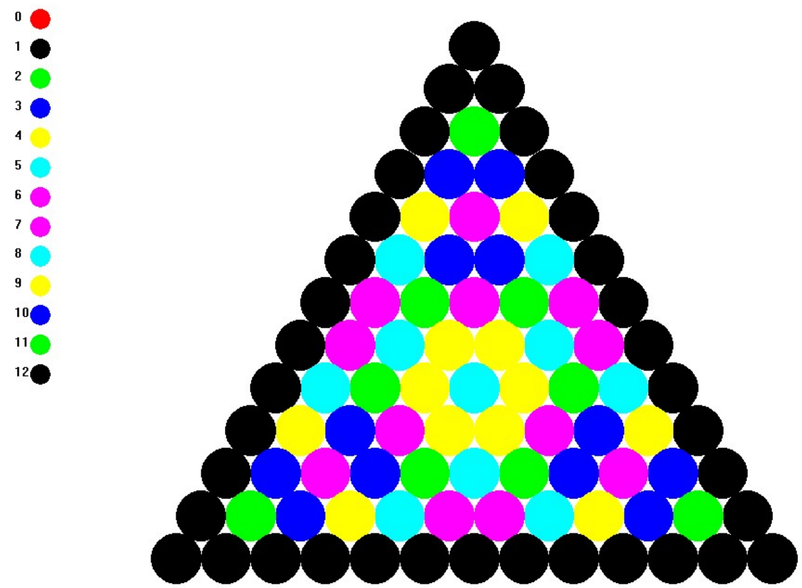


Conjecture. For a prime p and $1 \leq k \leq p-1$,

$$\sum_{j=0}^{k} \left[\begin{array}{c} p-1-k+j \\ j \end{array} \right] = 0.$$

Corollary. For a prime p, the sum of the first $p-2$ *triangular numbers* is zero modulo p. **Proof.** Take $k = p-3$.

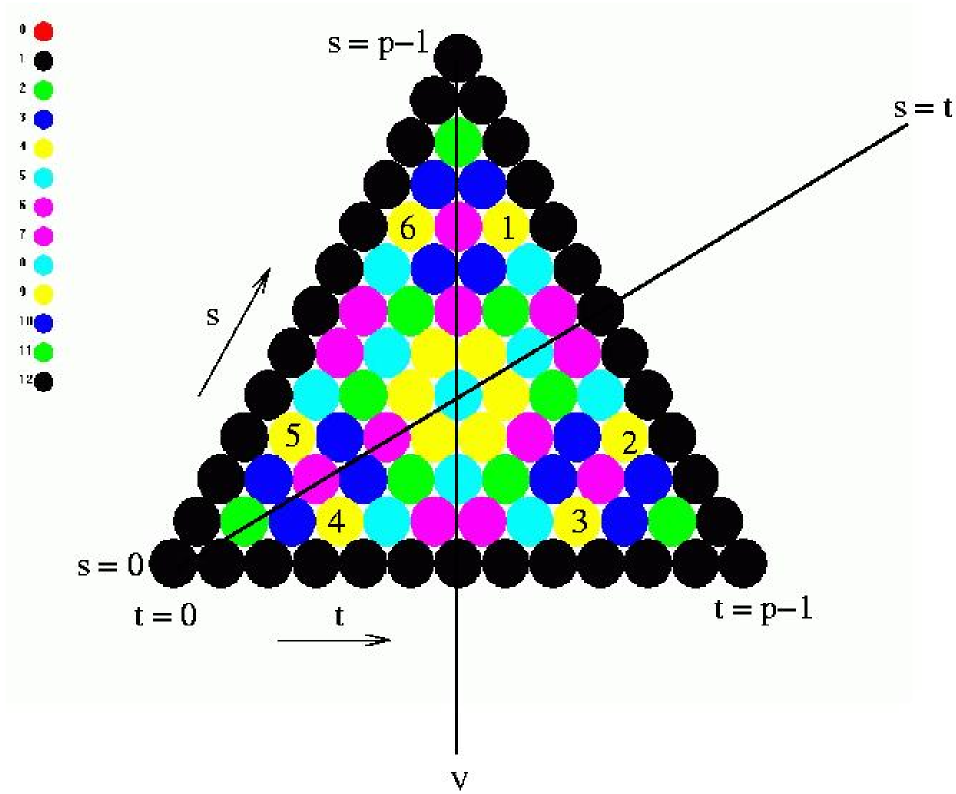
A More Complicated Symmetry



Generate $P_{\mathbb{Z}_p}(1,1)$ with $N = p-1$. Color elements $a \in \mathbb{Z}_p$ the same as their inverse $-a$.

What does this symmetry mean about $\pm \begin{bmatrix} n \\ k \end{bmatrix}$ for $0 \leq n \leq p-1$ and $0 \leq k \leq n$?

Translating the Picture (modulo all signs)

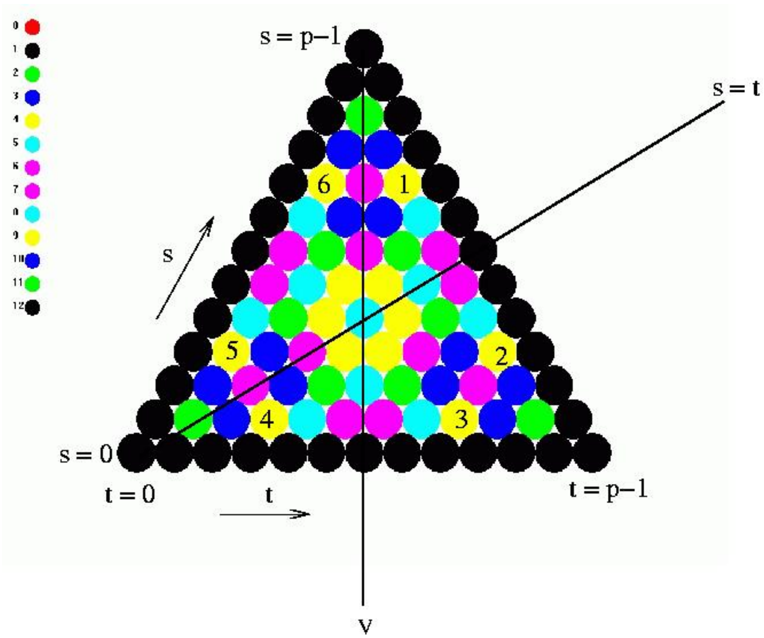


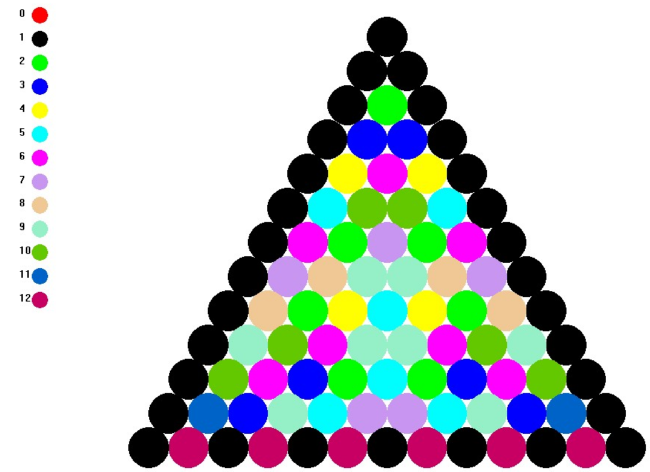
For $0 \leq s, t$, $0 \leq s+t \leq p-1$,

Symmetry in line V: $\left[\begin{array}{c} p-1-s \\ t \end{array} \right] = \left[\begin{array}{c} p-1-s \\ p-1-(s+t) \end{array} \right]$ $((1) = (6))$.

Symmetry in line $s = t$: $\left[\begin{array}{c} p-1-s \\ t \end{array} \right] = \pm \left[\begin{array}{c} p-1-t \\ s \end{array} \right]$ $((1) = (2))$.

Putting in the Signs





The reflection in the line $s = t$ changes the sign when $s + t$ is odd.

Conjecture. For p prime, $0 \leq s, t$, $0 \leq s + t \leq p - 1$,

$$\left[\begin{array}{c} p - 1 - s \\ t \end{array} \right] = (-1)^{s+t} \left[\begin{array}{c} p - 1 - t \\ s \end{array} \right].$$

Combining Results: The Remaining Reflection

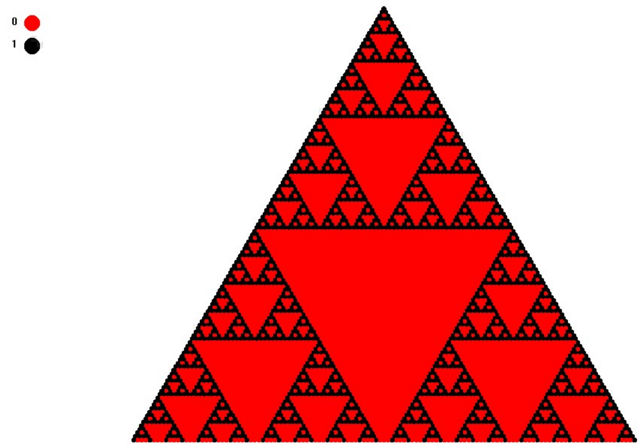
For a prime p and $0 \leq s+t \leq p-1$, $s,t \geq 0$:

$$\begin{aligned}
\left[\begin{array}{c} p-1-s \\ t \end{array} \right] &= (-1)^{s+t} \left[\begin{array}{c} p-1-t \\ s \end{array} \right] \\
&= (-1)^{s+t} \left[\begin{array}{c} p-1-t \\ p-1-(s+t) \end{array} \right] \\
&= (-1)^{t} \left[\begin{array}{c} s+t \\ t \end{array} \right]
\end{aligned}$$

Thank You!

PascalGT Software Download URL

http://faculty.salisbury.edu/~kmshannon/pascal/



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