Key

1. Differentiate the following functions. No simplifying. Take derivative and stop.

a. $f(x)=4^x-3x^4+\frac{2}{x^6}-e^{3x}+\log_4 x = 4^x-3x^4+2x^{-6}-e^{3x}+\log_4 x$

$$\boxed{f'(x)=\ln 4\cdot 4^x-12x^3-12x^{-7}-e^{3x}\cdot 3+\frac{1}{x\ln(4)}}$$

b. $h(x)=x6^{\sqrt{x}}$ ← product rule

$h(x)=x\cdot 6^{x^{1/2}}$

$$\boxed{h'(x)=x\cdot\ln(6)\cdot 6^{x^{1/2}}\cdot\frac{1}{2\sqrt{x}}+6^{x^{1/2}}\cdot 1}$$

c. $g(x)=3e^{5x^2+4x-1}$

$$\boxed{g'(x)=3\cdot e^{5x^2+4x-1}\cdot(10x+4)}$$

d. $f(x)=\ln(5x+\sqrt{x}+e)$

$$\boxed{f'(x)=\frac{1}{5x+\sqrt{x}+e}\cdot\left(5+\frac{1}{2\sqrt{x}}\right)}$$

Recall, $\frac{d}{dx}\ln(\text{stuff})=\frac{1}{\text{stuff}}\cdot\text{stuff}'$

e. $h(x)=e^{2x}\ln x$ ← product rule

$$\boxed{h'(x)=e^{2x}\cdot\frac{1}{x}+\ln x\cdot e^{2x}\cdot 2}$$

f. $g(t)=\frac{e^t-\sqrt{t}}{t+5^t}$ Quotient rule

$$\boxed{g'(t)=\frac{(t+5^t)\left(e^t-\frac{1}{2\sqrt{t}}\right)-(e^t-\sqrt{t})(1+\ln 5\cdot 5^t)}{(t+5^t)^2}}$$

2. Use the **second derivative test** to identify the relative maxima and/or minima.

$f(x) = x^4 - 2x^2 + 3$

$f'(x) = 4x^3 - 4x$

$0 = 4x(x^2 - 1)$

$0 = 4x(x+1)(x-1)$

$\boxed{x = 0,\ x = \pm 1}$ cp's

$f''(x) = 12x^2 - 4$

$f''(0) = -4 < 0$

(0,3) is a relative maximum

$f''(\pm 1) = 12(\pm 1)^2 - 4$
$= 12 - 4$
$= 8 > 0$

(1,2) & (-1,2) are relative minima

$f(\pm 1) = (\pm 1)^4 - 2(\pm 1)^2 + 3$
$= 1 - 2 + 3$
$= -1 + 3$
$= 2$

3. Find all vertical and horizontal asymptotes (if they exist) using CALCULUS techniques.

$f(x) = \dfrac{x+2}{x^2+5x+6} = \dfrac{x+2}{(x+3)(x+2)}$ $\quad x \neq -3, -2$

H.A. Take lim as $x \to \infty$

$\lim_{x \to \infty} \dfrac{x+2}{x^2+5x+6}$

$= \lim_{x \to \infty} \dfrac{\frac{x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{6}{x^2}}$ $\quad$ ($\to 0$, $\to 0$, $\to 0$, $\to 0$)

$= \dfrac{0}{1}$

$= 0$

Horizontal Asymptote @ y=0

Check x=-3

$\lim_{x \to -3^-} \dfrac{x+2}{x^2+5x+6}$

$= \lim_{x \to -3^-} \dfrac{\cancel{x+2}}{(\cancel{x+2})(x+3)}$

$= \lim_{x \to -3^-} \dfrac{1}{x+3}$

$= \dfrac{1}{-3.0001+3}$

$= \dfrac{1}{-\text{small}}$

$= -\infty$

Since $\lim_{x \to -3^-} f(x) = -\infty$, there is a vertical asymptote @ $x = -3$

Check x=-2

$\lim_{x \to -2^-} \dfrac{\cancel{x+2}}{(\cancel{x+2})(x+3)}$

$= \lim_{x \to -2^-} \dfrac{1}{x+3}$

$= \dfrac{1}{-2+3}$

$= \dfrac{1}{1}$

$= 1$

No V.A. @ x=-2

4. Find the extrema (**absolute** maxima and minima) of the function.

a. $f(x)=x^5-5x^4+1$ on the interval $0\le x\le 5$

$f'(x)=5x^4-20x^3$

$0=5x^3(x-4)$

$5x^3=0$ $\quad x-4=0$

$x^3=0$ $\quad x=4$

$x=0$

Endpoints

$x=0$

$f(0)=1$

$(0,1)$

$x=5$

$f(5)=5^5-5(5)^4+1$

$=3125-3125+1$

$=1$

$(5,1)$ ← max

Critical Points

$x=0$

$f(0)=1$

$(0,1)$

$x=4$

$f(4)=4^5-5(4)^4+1$

$=1024-5(256)+1$

$=1024-1280+1$

$=-256+1$

$=-255$

$(4,-255)$ ← min

The absolute minimum is at $(4,-255)$ and the absolute maximum is 1 and occurs at 0 and 5.

b. $f(x)=\left(x^2-4\right)^5$ on the interval $-3\le x\le 2$

$f'(x)=5(x^2-4)^4(2x)$

$f'(x)=10x(x^2-4)^4$

$0=10x(x^2-4)^4$

$10x=0$ $\quad (x^2-4)^4=0$

$x=0$ $\quad x^2-4=0$

$x^2=4$

$x=\pm\sqrt{4}$

$x=\pm 2$

Endpoints

$x=-3$

$f(-3)=((-3)^2-4)^5$

$=(9-4)^5$

$=5^5$

$=3125$

$(-3,3125)$ ← max

$x=2$

$f(2)=(2^2-4)^5$

$=0$

$(2,0)$

Critical Points

$x=0$

$f(0)=(-4)^5$

$=-1024$

$(0,-1024)$ ← min

$x=-2$

$f(-2)=((-2)^2-4)^5$

$=0$

$(-2,0)$

The absolute maximum is at $(-3, 3{,}125)$ and the absolute minimum is at $(0,-1{,}024)$.

5. Let $f(x) = \dfrac{x}{x^2+1}$

a. Find the **x and y intercepts** of the function if they exist. Answer as ordered pairs.

y int $\Rightarrow x = 0$

$f(0) = \dfrac{0}{0+1} = 0$

$(0,0)$

x int $\Rightarrow y = 0$

$0 = \dfrac{x}{x^2+1}$

$x = 0$

$(0,0)$

The x-int & y-int is $(0,0)$.

b. Find the **equations of any vertical and horizontal asymptotes**, if they exist. You may use precalculus reasoning, or calculus techniques.

There is no place where $f(x)$ is undefined since x^2+1 will never equal zero, so there are no vertical asymptotes.

There is a horizontal asymptote at $\boxed{y=0}$ since the degree of the denom is greater than the degree of the numerator.

c. Using calculus (use first or second derivative test), identify **relative maxima and minima**, if they exist.

$f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$

$= \dfrac{x^2+1-2x^2}{(x^2+1)^2}$

$f'(x) = \dfrac{-x^2+1}{(x^2+1)^2}$

$0 = -x^2+1$

$-1 \qquad -1$

$-1 = -x^2$

$x^2 = 1$

$x = \pm 1$

f' $\;- - - - \,|\, + + + + + + \,|\, - - - -$

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

Test points

$f'(-2) = \dfrac{-(-2)^2+1}{\text{(Always positive)}}$

$= -4+1 < 0$

$f'(0) = 0+1 > 0$

$f'(2) = -(2)^2+1$

$= -4+1 < 0$

Y values

$f(1) = \dfrac{1}{1^2+1} = \dfrac{1}{2}$

$f(-1) = \dfrac{-1}{1+1} = \dfrac{-1}{2}$

There is a relative max @ $(1, \frac{1}{2})$ and a relative min @ $(-1, -\frac{1}{2})$.

Recall, $f(x) = \dfrac{x}{x^2+1}$

d. Find the **inflection point(s),** if any, and the **intervals** where the function is concave up and concave down.

$$f'(x) = \frac{-x^2+1}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^2(-2x) - (-x^2+1)\left(2(x^2+1)'\cdot 2x\right)}{(x^2+1)^4}$$

$$= \frac{\cancel{(x^2+1)}\left[(x^2+1)(-2x) - 4x(-x^2+1)\right]}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{-2x^3-2x+4x^3-4x}{(x^2+1)^3}$$

$$= \frac{2x^3-6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$ Always positive

$0 = 2x(x^2-3)$

$2x = 0 \quad x^2-3=0$

$x = 0 \quad x^2 = 3$

$x = \pm\sqrt{3} \approx \pm 1.73$

f'': $- - - -$ (concave down) | $-\sqrt{3}$ | $+ + + + +$ (concave up) | 0 | $- - - -$ (concave down) | $\sqrt{3}$ | $+ + + + +$ (concave up)

$-2 \quad -\sqrt{3} \quad -1 \quad 0 \quad 1 \quad \sqrt{3} \quad 2$

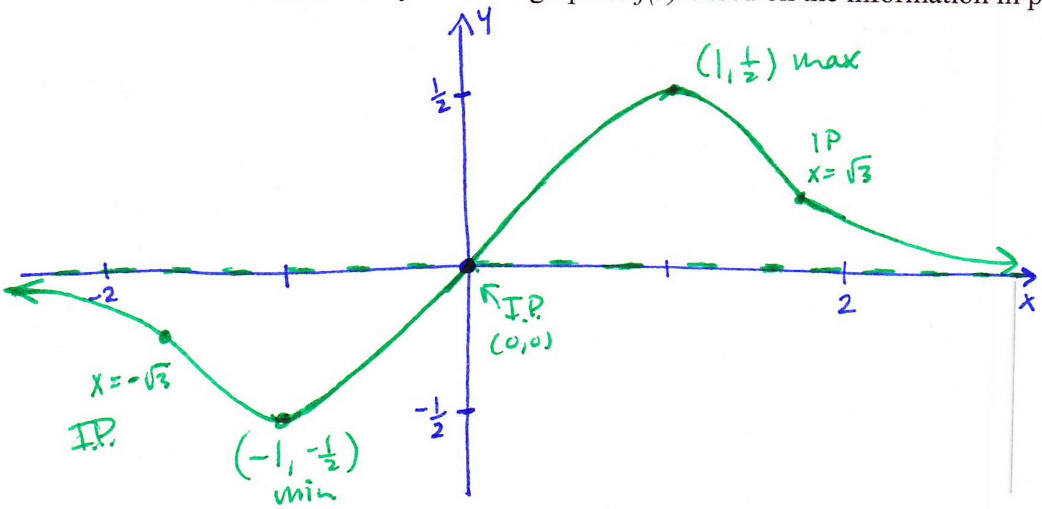
$f''(-2) = 2(-2)((-2)^2-3) = -(+) = -$

$f''(-1) = 2(-1)((-1)^2-3) = - \cdot - = +$

$f''(1) = 2(1)(1^2-3) = + \cdot - = -$

$f''(2) = 2(2)(2^2-3) = + \cdot + = +$

e. Sketch a fairly accurate graph of *f(x)* based on the information in parts a-d.



$\boxed{\begin{array}{c}\text{Concave up} \\ (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty) \\ \text{Concave down} \\ (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}) \\ \text{Inflection Pts @} \\ x = \pm\sqrt{3},\ 0\end{array}}$

6. Find the equation of the tangent line at the specific point.

$f(x) = (x+1)e^{-2x}$; where $x = 0$

product rule

$f'(x) = (x+1)e^{-2x}(-2) + e^{-2x}(1)$

$= e^{-2x}\left(-2(x+1)+1\right)$

$= e^{-2x}(-2x-2+1)$

$= e^{-2x}(-2x-1)$

$f'(0) = e^{-2\cdot 0}(-2(0)-1)$

$= e^0(-1)$

$= 1 \cdot -1 = -1$ ← slope of the tangent line @ $x=0$

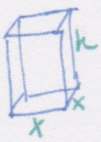
$m = -1 \quad (0, 1)$ ← y int

$f(0) = 1e^0 = 1$

$\boxed{y = -1x + 1}$

7. A carpenter has been asked to build an open box with a square base. You want to box to have a volume of 200 cubic meters. What are the dimensions of the box that will use the least amount of material? (Also study Sections 1.4 and 3.5 homework and lecture notes examples :)

minimize surface area!



$\boxed{x^2} \times 1$

$\boxed{xh} \times 4$

(A) $S.A. = x^2 + 4xh$

$= x^2 + 4x\left(\frac{200}{x^2}\right)$

$S(x) = x^2 + \frac{800}{x} = x^2 + 800x^{-1}$

$S'(x) = 2x + (-800)x^{-2}$

$0 = 2x - 800x^{-2}$

$\frac{800}{x^2} = 2x$

$\frac{800}{2} = \frac{2x^3}{2}$

$x^3 = 400$ $\rightarrow$ $x = \sqrt[3]{400}$ $\rightarrow$ $h = \frac{200}{(\sqrt[3]{400})^2}$

(B) Volume $= l \cdot w \cdot h$

$200 = x \cdot x \cdot h$

$200 = x^2 \cdot h$

$\frac{200}{x^2} = h$

The base would measure $\sqrt[3]{400}$ meters and the hight would be $200/(\sqrt[3]{400})^2$ meters.

8. Population Growth. It is projected that t years from now, the population of a certain town will be approximately P(t) thousand people, where $P(t) = \frac{100}{1+e^{-0.2t}}$

a. What is the current population?

b. At what rate will the population be changing 10 years from now? $P'(t)$

a) current pop $\Rightarrow$ $t = 0$

$P(0) = \frac{100}{1+e^0}$

$= \frac{100}{1+1}$

$= \frac{100}{2}$

$= 50$

The current population is 50 thousand people.

b) $P'(t) = \frac{(1+e^{-0.2t}) \cdot 0 - 100(e^{-0.2t} \cdot -0.2)}{(1+e^{-0.2t})^2}$

$P'(t) = \frac{20e^{-0.2t}}{(1+e^{-0.2t})^2}$

$P'(10) = \frac{20e^{-0.2(10)}}{(1+e^{-0.2(10)})^2}$

$= \frac{20e^{-2}}{(1+e^{-2})^2}$

≈ 2.09987

The rate of change of the population is about 2.1 thousand people per year.