Key

Practice Exam #1

Math 105 - Fall 2012

Show as much work as possible. Answers with no work shown will receive no credit.

1. Calculate the following limits. For full credit, show your work.

Always try to plug in values FIRST!

a. $\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$

$= \lim_{x\to 3} x+3 = \boxed{6}$

mult by conjugate

b. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{(\sqrt{x}-2)}{x-4}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)$

Next FOIL numerator

$= \lim_{x\to 4}\frac{x+\cancel{2\sqrt{x}}-\cancel{2\sqrt{x}}-4}{(x-4)(\sqrt{x}+2)}$

$= \lim_{x\to 4}\frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x}+2)} = \lim_{x\to 4}\frac{1}{\sqrt{x}+2}$

$= \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

c. $\lim_{x\to\infty}\frac{2x^3-x-3}{6x^2-x-1}$

Divide by highest power in denom

$= \lim_{x\to\infty}\frac{\frac{2x^3}{x^2}-\frac{x}{x^2}-\frac{3}{x^2}}{\frac{6x^2}{x^2}-\frac{x}{x^2}-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{2x-\cancel{\frac{1}{x}}^{\to 0}-\cancel{\frac{3}{x^2}}^{\to 0}}{6-\cancel{\frac{1}{x}}^{\to 0}-\cancel{\frac{1}{x^2}}^{\to 0}} = \lim_{x\to\infty}\frac{2x}{6}$

$= \boxed{\infty}$

d. $\lim_{x\to\infty}\frac{2x^3-x-3}{6x^3-x-1}$

$= \lim_{x\to\infty}\frac{\frac{2x^3}{x^3}-\frac{x}{x^3}-\frac{3}{x^3}}{\frac{6x^3}{x^3}-\frac{x}{x^3}-\frac{1}{x^3}} = \lim_{x\to\infty}\frac{2-\cancel{\frac{1}{x^2}}^{\to 0}-\cancel{\frac{3}{x^3}}^{\to 0}}{6-\cancel{\frac{1}{x^2}}^{\to 0}-\cancel{\frac{1}{x^3}}^{\to 0}}$

$= \frac{2}{6} = \boxed{\frac{1}{3}}$

e. $\lim_{x\to\infty}\frac{2x^2-x-3}{6x^3-x-1}$

$\lim_{x\to\infty}\frac{\frac{2x^2}{x^3}-\frac{x}{x^3}-\frac{3}{x^3}}{\frac{6x^3}{x^3}-\frac{x}{x^3}-\frac{1}{x^3}} = \lim_{x\to\infty}\frac{\cancel{\frac{2}{x}}^{\to 0}-\cancel{\frac{1}{x^2}}^{\to 0}-\cancel{\frac{3}{x^3}}^{\to 0}}{6-\cancel{\frac{1}{x^2}}^{\to 0}-\cancel{\frac{1}{x^3}}^{\to 0}}$

$= \frac{0}{6} = \boxed{0}$

2. Evaluate the following limits. Use proper limit notation.

$-0.001 \nearrow 0 \nwarrow 0.001$

a. $\lim_{x\to 0^+} \frac{1}{x} \approx \frac{1}{+0.001}$

$\approx \frac{1}{+\text{small}}$

$\approx \boxed{\infty}$

Recall,

$\frac{\pm \#}{\pm \text{small}} = \pm \text{BIG}$

$\frac{\pm \#}{\pm \text{BIG}} = \pm \text{small}$

b. $\lim_{x\to 0^-} \frac{1}{x}$

$\approx \frac{1}{-0.001} \approx \frac{1}{-\text{small}}$

$\approx \boxed{-\infty}$

c. $\lim_{x\to 0} \frac{1}{x} = \boxed{\text{DNE}}$ Since left & right hand limits are not equal.

$\lim_{x\to 0^-} \frac{1}{x} \neq \lim_{x\to 0^+} \frac{1}{x}$ *!

3. Use left and right hand limits and proper limit notation to evaluate the following limits.

$4.999 \nearrow 5$

a. $\lim_{x\to 5^-} \frac{-3x}{x-5} \approx \frac{-3(4.999)}{4.999-5}$

$\approx \frac{-15}{-\text{small}}$

$\approx \boxed{\infty}$

b. $\lim_{x\to 1^-} \frac{x-\sqrt{x}}{x-1} \left(\frac{x+\sqrt{x}}{x+\sqrt{x}}\right)$ ← FOIL

$= \lim_{x\to 1^-} \frac{x^2 - x}{(x-1)(x+\sqrt{x})}$

$= \lim_{x\to 1^-} \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+\sqrt{x})}$

$= \lim_{x\to 1^-} \frac{x}{x+\sqrt{x}}$

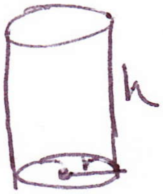
$= \frac{1}{1+\sqrt{1}}$

$= \boxed{\frac{1}{2}}$

②

2. A closed cylindrical can has surface area 120π square inches. Express the volume of the can as a function of its radius. Be sure to declare all variables. (Recall, volume of a cylinder $V = \pi r^2 h$)

Goal $V(r)$



$\pi r^2 \times 2$

$2\pi r$

$2\pi r h$ h

Ⓐ $V = \pi r^2 h \Longrightarrow$ Ⓑ $S.A. = 2\pi r^2 + 2\pi r h$

$120\pi = 2\pi r^2 + 2\pi r h$

$-2\pi r^2 \quad -2\pi r^2$

$\dfrac{120\pi - 2\pi r^2}{2\pi r} = \dfrac{2\pi r h}{2\pi r}$

$h = \dfrac{120\pi - 2\pi r^2}{2\pi r}$

$h = \dfrac{\cancel{2\pi}(60 - r^2)}{\cancel{2\pi} r}$

$h = \dfrac{60 - r^2}{r}$

substitute into

$V = \pi r^2 \left(\dfrac{60 - r^2}{r}\right)$

$V = \pi r(60 - r^2)$

$\boxed{V(r) = \pi r(60 - r^2)}$

OR

$V(r) = 60\pi r - \pi r^3$

3. Consider the function $f(x) = \dfrac{-3}{x+2}$.

a. What's the domain of this function (use interval or set notation)?

$\boxed{\text{Domain} = \{X \mid X \neq -2\}}$

Denom $\neq 0$

$X + 2 \neq 0 \quad X \neq -2$

b. Does the function have a horizontal asymptote? If so, what's its equation?

There is a horizontal asymptote at $\boxed{y=0}$ since the degree of the denom is greater than degree of numerator.

Okay here to use precalc knowledge

c. Does the function have a y-intercept? If so, find it.

y-int $\Rightarrow X = 0$

$f(0) = \dfrac{-3}{0+2} = -\dfrac{3}{2}$ $\boxed{(0, -\tfrac{3}{2}) \text{ y-intercept}}$ ← ordered pair!

d. Is f continuous at $x = 2$? If so, show all parts of the definition of continuity hold. If not, show at least one fails.

$X = 2$

① $f(2) = \dfrac{-3}{2+2} = \dfrac{-3}{4}$ ✓

② $\lim\limits_{X \to 2} f(x) = \lim\limits_{X \to 2} \dfrac{-3}{X+2} = \dfrac{-3}{2+2} = -\dfrac{3}{4}$ ✓

OR $=$

$\lim\limits_{x \to 2} f(x) = \dfrac{-3}{4} = f(2)$

③ ① & ② are equal so yes $\boxed{f(x) \text{ is cont at } x = 2.}$

$$f(x) = \frac{-3}{x+2}$$

3

e. Is f continuous at $x = -2$? If so, show all parts of the definition of continuity hold. If not, show at least one fails.

$$① \quad f(-2) = -\frac{3}{-2+2} = -\frac{3}{0} = \boxed{\text{undefined}}$$

$\boxed{\text{Since ① Fails, } f(x) \text{ is Not continuous at } x=-2}$

f. Calculate the derivative of f using the definition of the derivative. (For full credit, use the "long" method we learned in class)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{-3}{x+h+2} + \frac{+3}{x+2}}{h} = \lim_{h\to 0} \frac{\frac{-3}{x+h+2}\left(\frac{x+2}{x+2}\right) + \frac{3}{x+2}\left(\frac{x+h+2}{x+h+2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-3(x+2)+3(x+h+2)}{(x+h+2)(x+2)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-3x}\cancel{-6}+\cancel{3x}+3h+\cancel{6}}{h(x+h+2)(x+2)} = \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{3}{(x+h+2)(x+2)} = \frac{3}{(x+0+2)(x+2)} = \boxed{\frac{3}{(x+2)^2}}$$

g. Write the equation of the tangent line to f at the point $(1, -1)$. Please give the equation in slope-intercept form.

$x = 1$

$$f'(x) = \frac{3}{(x+2)^2} \Rightarrow f'(1) = \frac{3}{(1+2)^2} = \frac{3}{3^2} = \frac{1}{3} = m$$

$$m = \frac{1}{3} \qquad (\underset{x_1}{1}, \underset{y_1}{-1}) \longrightarrow y - y_1 = m(x - x_1)$$

$$y + 1 = \frac{1}{3}(x-1)$$

$$y + 1 = \frac{1}{3}x - \frac{1}{3}$$

$$y = \frac{1}{3}x - \frac{1}{3} - \frac{3}{3}$$

$$\boxed{y = \frac{1}{3}x - \frac{4}{3}}$$

h. What's the **average rate of change** of f between $x = 1$ and $x = 3$?

$a \qquad b$

$$\text{Ave. RoC} = \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{-\frac{3}{5}-(-1)}{2} = \frac{-\frac{3}{5}+\frac{5}{5}}{2} = \frac{\frac{2}{5}}{2} = \frac{2}{5}\cdot\frac{1}{2} = \boxed{\frac{1}{5}}$$

$\boxed{\text{The average RoC. is } \frac{1}{5}.}$

Sidework

$$f(3) = \frac{-3}{3+2} = \frac{-3}{5}$$

$$f(1) = \frac{-3}{1+2} = \frac{-3}{3} = -1$$

Find difference Quotient 1st then take limit

Difference Quotient $\frac{f(x+h)-f(x)}{h}$ (4)

4. Use the definition of the derivative to compute the following derivatives. Simplify all answers completely.

a. $f(x) = 2 - 7x$

$$f'(x) = \lim_{h \to 0} -7 = \boxed{-7}$$

Difference Quotient

$$a)\ \frac{2-7(x+h)-(2-7x)}{h}$$

$$= \frac{\cancel{2}\cancel{-7x}-7h\cancel{-2}\cancel{+7x}}{h}$$

$$= \frac{-7\cancel{h}}{\cancel{h}} = -7$$

b. $f(x) = 5x^2$

$$f'(x) = \lim_{h \to 0} 10x + 5h = 10x + 5(0)$$

$$= \boxed{10x}$$

$$b)\ \frac{5(x+h)^2 - 5x^2}{h}$$

$$= \frac{5(x+h)(x+h) - 5x^2}{h}$$

$$= \frac{5(x^2+2xh+h^2) - 5x^2}{h}$$

$$= \frac{\cancel{5x^2}+10xh+5h^2\cancel{-5x^2}}{h}$$

$$= \frac{10xh+5h^2}{h}$$

$$= 10x + 5h$$

Another method: OR you could write it all at once

c. $f(x) = 2x^2 + 4x - 1$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + 4(x+h) - 1 - (2x^2 + 4x - 1)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)(x+h) \cancel{+4x} + 4h \cancel{-1} - 2x^2 \cancel{-4x} \cancel{+1}}{h}$$

$$= \lim_{h \to 0} \frac{2(x^2+2xh+h^2) + 4h - 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + 4h \cancel{-2x^2}}{h} = \lim_{h \to 0} \frac{4xh + 2h^2 + 4h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h + 4)}{\cancel{h}} = \lim_{h \to 0} 4x + 2h + 4 = 4x + 2(0) + 4$$

$$= \boxed{4x + 4}$$

d. $f(x) = \frac{6}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{6}{x+h} - \frac{6}{x}}{h} = \lim_{h\to 0} \frac{\frac{6}{x+h}\left(\frac{x}{x}\right) - \frac{6}{x}\left(\frac{x+h}{x+h}\right)}{h}$$

$$= \lim_{h\to 0} \left[\frac{6x}{(x+h)(x)} - \frac{6(x+h)}{(x+h)(x)}\right] \cdot \frac{1}{h} = \lim_{h\to 0} \left[\frac{6x - 6(x+h)}{(x+h)\cdot x}\right] \frac{1}{h}$$

$$= \lim_{h\to 0} \left[\frac{\cancel{6x} - \cancel{6x} - 6h}{x(x+h)} \cdot \frac{1}{h}\right] = \lim_{h\to 0} \frac{-6\cancel{h}}{x\cdot \cancel{h} \cdot (x+h)} = \lim_{h\to 0} \frac{-6}{x(x+h)}$$

$$= \frac{-6}{x(x+0)} = \boxed{\frac{-6}{x^2}}$$

e. $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{\cancel{x}+h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

5. Find the **equation of the tangent line** in slope intercept form that is tangent to $f(x) = x^2$ at $x = 3$.

(Hint: Use the definition of the derivative to compute the slope, then find the y value for the given x value. Now you have a slope and a point and can compute the equation of this tangent line.)

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{(x+h)(x+h) - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h = 2x + 0 = 2x \longrightarrow f'(x) = 2x$$

$$f'(3) = 2(3) = 6 = m \qquad (3, 9) \qquad y - 9 = 6(x - 3)$$

$$y = 6x - 18 + 9$$

$$\boxed{y = 6x - 9}$$