Math 105 - Bio-Calculus

Exam 1

- ✓ You have 50 minutes to take the exam.
- ✓ Make sure you use **concepts/techniques from calculus** in order to answer the questions.
- ✓ You must SHOW ALL WORK and simplify all answers completely unless otherwise stated in order to receive full credit.
- ✓ Please indicate your answers by circling or boxing them.
- ✓ You may **NOT** use any notes, book, or neighbors during the exam.
- ✓ You are allowed to use a calculator on the exam.
- ✓ Leave all answers **EXACT (no decimals) and use improper fractions** when necessary.
- ✓ If you feel that you may be on the wrong track, put an x through work and try problem over on scratch paper. Many times, you are on the right track, but second-guess yourself.

I have read the above guidelines and agree to follow them. Also, the work contained on this exam is my own and I promise to adhere to academic honesty.

Name: Key

Signature: ______________________________

Good Luck!

SHOW AS MUCH WORK AS POSSIBLE. Answers with no work shown will receive no credit. Feel free to use sentences to explain answers as well if needed to accompany your work.

1. Find the **average rate of change** of the function between $\underset{a}{x=-4}$ and $\underset{b}{x=5}$

$f(x)=3x^2-2x$

$$\text{Ave R.O.C.} = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{f(5)-f(-4)}{5-(-4)}$$

$$= \frac{65-56}{5+4} = \frac{9}{9} = \boxed{1}$$

$$f(5)=3(5)^2-2(5) \\ =3(25)-10 \\ =75-10 \\ =65$$

$$f(-4)=3(-4)^2-2(-4) \\ =3(16)+8 \\ = 48+8 \\ =56$$

2. Calculate the following limits. For full credit show all work.

a. $\lim_{x\to 49}\left(\frac{\sqrt{x}-7}{x-49}\right)\left(\frac{\sqrt{x}+7}{\sqrt{x}+7}\right)$ ← FOIL

$$= \lim_{x\to 49} \frac{\cancel{x-49}\ 1}{\cancel{(x-49)}(\sqrt{x}+7)}$$

$$= \lim_{x\to 49} \frac{1}{\sqrt{x}+7}$$

$$= \frac{1}{\sqrt{49}+7}$$

$$= \frac{1}{7+7}$$

$$= \boxed{\frac{1}{14}}$$

b. $\lim_{x\to -4}\frac{x^2-2x-24}{2x^2+11x+12}$

Try to plug in -4

$2(-4)^2+11(-4)+12$

$2(16)-44+12$

$32-44+12$

$44-44$

0

division by zero.

So factor & reduce!

Factoring

$2x^2+11x+12$

$2x^2+8x+3x+12$

$2x(x+4)+3(x+4)$

$(x+4)(2x+3)$

$$= \lim_{x\to -4} \frac{(x-6)\cancel{(x+4)}}{\cancel{(x+4)}(2x+3)}$$

$$= \lim_{x\to -4} \frac{x-6}{2x+3}$$

$$= \frac{-4-6}{2(-4)+3} = \frac{-10}{-8+3} = \frac{-10}{-5} = \boxed{2}$$

Divide by highest power of x in denom.

3. Calculate the following limits at infinity. Use algebraic techniques/concepts from calculus.

a. $\lim_{x \to -\infty} \frac{3x^5 - 4x^2 + 7x}{9x^3 + 3x^4 - 14} \cdot \left(\frac{\frac{1}{x^4}}{\frac{1}{x^4}}\right)$

$= \lim_{x \to -\infty} \frac{\frac{3x^5}{x^4} - \frac{4x^2}{x^4} + \frac{7x}{x^4}}{\frac{9x^3}{x^4} + \frac{3x^4}{x^4} - \frac{14}{x^4}}$

$= \lim_{x \to -\infty} \frac{3x - \cancel{\frac{4}{x^2}}^{\to 0} + \cancel{\frac{7}{x^3}}^{\to 0}}{\cancel{\frac{9}{x}}^{\to 0} + 3 - \cancel{\frac{14}{x^4}}^{\to 0}}$

$= \lim_{x \to -\infty} \frac{\cancel{3}x}{\cancel{3}}$

$= \lim_{x \to -\infty} x = \boxed{-\infty}$

b. $\lim_{x \to \infty} \frac{6x^4 - 2x^3 + 9x}{7 - 6x^2 - 8x^4} \cdot \left(\frac{\frac{1}{x^4}}{\frac{1}{x^4}}\right)$

$= \lim_{x \to \infty} \frac{\frac{6x^4}{x^4} - \frac{2x^3}{x^4} + \frac{9x}{x^4}}{\frac{7}{x^4} - \frac{6x^2}{x^4} - \frac{8x^4}{x^4}}$

$= \lim_{x \to \infty} \frac{6 - \cancel{\frac{2}{x}}^{\to 0} + \cancel{\frac{9}{x^3}}^{\to 0}}{\cancel{\frac{7}{x^4}}^{\to 0} - \cancel{\frac{6}{x^2}}^{\to 0} - 8}$

$= \frac{6}{-8}$

$= \boxed{-\frac{3}{4}}$

Recall

$\frac{\pm \#}{\pm \text{small}} = \pm \text{BIG}$

$\frac{\pm \#}{\pm \text{BIG}} = \pm \text{small}$

4. Compute the following limit by calculating the two one-sided limits. SHOW WORK!

$\lim_{x \to 6} \frac{-7x}{x-6}$

5.999 → 6 ← 6.0001

$\lim_{x \to 6^-} \frac{-7x}{x-6} \approx \frac{-7(5.999)}{5.999-6}$

$\approx \frac{-42}{-\text{small}}$

$\approx \boxed{\infty}$

$\lim_{x \to 6^+} \frac{-7x}{x-6} = \frac{-7(6.0001)}{6.0001-6}$

$= \frac{-42}{+\text{small}}$

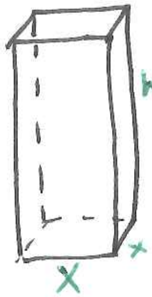
$= \boxed{-\infty}$

In general, the two sided limit

$\lim_{x \to 6} \frac{-7x}{x-6} = \boxed{\text{DNE}}$

Since $\lim_{x \to 6^-} f(x) \neq \lim_{x \to 6^+} f(x)$.

5. A closed box with a square base has a surface area of 1,600 square centimeters. Express its **volume** as a function of the length of its base (Recall, volume of a box is $V = l \cdot w \cdot h$)



Ⓐ Function $V(x) = ?$

Volume $= l \cdot w \cdot h$

$= x \cdot x \cdot h$

$= x^2 h$ ← Substitute

$$V(x) = x^{\cancel{2}}\left(\frac{800-x^2}{2\cancel{x}}\right)$$

$$\boxed{V(x) = x\left(\frac{800-x^2}{2}\right)}$$

OR

$$V(x) = \frac{800x - x^3}{2}$$

OR $V(x) = 400x - \frac{1}{2}x^3$

Ⓑ Equation

S.A. $= 1600$

$4xh + 2x^2 = 1600$ solve for h

$4xh = 1600 - 2x^2$

$$h = \frac{1600 - 2x^2}{4x}$$

$$h = \frac{\cancel{2}(800 - x^2)}{{}_{2}\cancel{4}x}$$

$$h = \boxed{\frac{800 - x^2}{2x}}$$

$h \; \boxed{xh} \; \times 4$ (width x)

$x \; \boxed{x^2} \; \times 2$ (width x)

6.Consider the function $f(x) = \frac{3}{x}$.

a. What's the domain of this function (use interval or set notation)?

$$\boxed{\begin{aligned} \text{Domain} &= \{x \mid x \neq 0\} \\ &= (-\infty, 0) \cup (0, \infty) \end{aligned}}$$

b. Does the function have a horizontal asymptote? If so, explain why, and **state its equation.** Either explain using concepts from ~~pre~~algebra, or use calculus techniques.

Algebra

Since degree of denom. is greater than degree of numerator, there is an HA. @ $\boxed{y = 0.}$

Calculus

$$\lim_{x \to \infty} \frac{3}{x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\frac{3}{x}}{1} = \frac{0}{1} = 0$$

Since, $\lim_{x \to \infty} \frac{3}{x}$ is 0, there is an HA @ $y = 0$

c. Is f continuous at $x = 1$? If so, show all parts of the definition of continuity hold. If not, show at least one fails.

$x = 1 \leftarrow c = 1$

① $f(c)$ exists

$f(x) = \frac{3}{x}$

$f(1) = \frac{3}{1}$

$= 3$ ✓

② $\lim_{x \to c} f(x)$ exists

$\lim_{x \to 1} \frac{3}{x}$

$= \frac{3}{1}$

$= 3$ ✓

③ $\lim_{x \to c} f(x) = f(c)$

$\lim_{x \to 1} \frac{3}{x} = f(1) = 3$ ✓

OR

① & ② are equal.

Yes, f(x) is continuous at $x = 1$.

Recall, $f(x) = \frac{3}{x}$

d. Calculate the derivative of f using the definition of the derivative. (For full credit, use the "long" method we learned in class)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{x+h}\left(\frac{x}{x}\right) - \frac{3}{x}\left(\frac{x+h}{x+h}\right)}{h}$$ mult by reciprocal

$$= \lim_{h\to 0} \frac{3x - 3(x+h)}{x(x+h)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x} - \cancel{3x} - 3h}{x(x+h)\cdot h}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{x\cancel{h}(x+h)}$$

$$= \lim_{h\to 0} \frac{-3}{x(x+h)}$$

$$= \frac{-3}{x(x+0)}$$

$$= \frac{-3}{x^2}$$

$$\boxed{f'(x) = -\frac{3}{x^2}}$$

e. Write the **equation** of the tangent line to f at the point $(-3, -1)$. Show work, and please give the equation in **slope-intercept form**. $x = -3$ x_1 y_1

Slope $= f'(-3)$

$$= \frac{-3}{(-3)^2}$$

$$= \frac{-3}{9}$$

$$m = -\frac{1}{3}$$

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = -\frac{1}{3}(x - (-3))$$

$$y + 1 = -\frac{1}{3}(x+3)$$

$$y + 1 = -\frac{1}{3}x - 1$$

$$-1 \qquad\qquad -1$$

$$\boxed{y = -\frac{1}{3}x - 2}$$

7. **Use the definition of the derivative** to compute the following derivatives. Simplify all answers completely. NO SHORTCUT RULES.

a. $f(x) = 3x + 5$

$f(x) = 3x+5$

$f(x+h) = 3(x+h)+5$

$= 3x+3h+5$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3x+3h+5-(3x+5)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x}+3h+\cancel{5}-\cancel{3x}-\cancel{5}}{h}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}} = \lim_{h\to 0} 3 = 3$$

$$\boxed{f'(x) = 3}$$

b. $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Rationalize numerator w/ conjugate

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

7 con't) **Use the definition of the derivative** to compute the following derivatives. Simplify all answers completely. NO SHORTCUT RULES.

c. $f(x) = -3x^2 + 4x$

$f(x) = -3x^2 + 4x$

$f(x+h) = -3(x+h)^2 + 4(x+h)$

$= -3(x+h)(x+h) + 4x + 4h$

$= -3(x^2 + 2xh + h^2) + 4x + 4h$

$= -3x^2 - 6xh - 3h^2 + 4x + 4h$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-3x^2 - 6xh - 3h^2 + 4x + 4h - (-3x^2 + 4x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-3x^2} - 6xh - 3h^2 \cancel{+4x} + 4h \cancel{+3x^2} \cancel{-4x}}{h}$$

$$= \lim_{h \to 0} \frac{-6xh - 3h^2 + 4h}{h}$$ factor & reduce!

$$= \lim_{h \to 0} \frac{\cancel{h}(-6x - 3h + 4)}{\cancel{h}}$$

$$= \lim_{h \to 0} -6x - 3h + 4$$

$= -6x - 3(0) + 4$

$= -6x + 4$

$\boxed{f'(x) = -6x + 4}$

8.. Find the **equation of the tangent line** in slope intercept form that is tangent to $f(x) = x^2$ at $x = -3$. Show steps and use the limit definition of the derivative.

$f(x) = x^2$

$f(x+h) = (x+h)^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$ factor & reduce

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h$$

$= 2x + 0$

$f'(x) = 2x$

slope!

$f'(-3) = 2(-3) = -6$

$m = -6 \quad (-3, f(-3)) = (\underset{x_1}{-3}, \underset{y_1}{9})$

$y - 9 = -6(x - (-3))$

$y - 9 = -6(x + 3)$

$y - 9 = -6x - 18$
$\quad +9 \qquad\qquad +9$

$\boxed{y = -6x - 9}$