$2^{-1} = \frac{1}{2}$

5.3 The Definite Integral & the Fundamental Theorem of Calculus

Known rate of change, but unknown function
→ Integrate with bounds.

Examples from last section:
- Distance traveled during the second minute
- Concent. of a drug over the 1st 4 hrs
- Change in biomass between 2nd & 5th hrs.
- Investments value change over time
- Known rate of water pollution rate of change to find pollution in next two years.
- Value of land (Area)

skip if low on time

Scenario

Suppose a real estate agent wants to evaluate unimproved land that is 100 ft wide bound by streets on 3 sides, and a stream on the 4th. It is determined the shape of the stream can be described by $y = x^3 + 1$ where x & y are in hundreds of feet.

Area of parcel A and land

Fundamental Theorem of Calculus

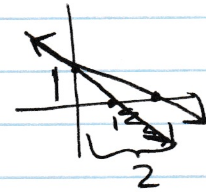
If $f(x)$ is continuous on $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Gives area under curve $f(x)$ from $a \leq x \leq b$.

where F is any antiderivative of $f(x)$ on $a \leq x \leq b$.

ex. $\int -\frac{1}{2}x + 1\,d$



$A = \frac{1}{2} b \cdot h$
$= \frac{1}{2} \cdot 2 \cdot 1$
$= 1$

ex. $\int_{-1}^{0} x^3\,dx =$ $\quad -\frac{1}{4}$

Rules

5.3 ex. $\int_{4}^{8}(2x+1)\,dx =$

Rules

Let f & g be continuous on $a \le x \le b$

then

1. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

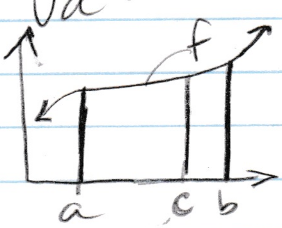
3. $\int_a^a f(x)\,dx = 0$

4. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ ← see pg 405 for justification

5. Subdivision Rule

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b g(x)\,dx$$

for $a \le c \le b$



Let f & g be continuous on $-2 \le x \le 5$

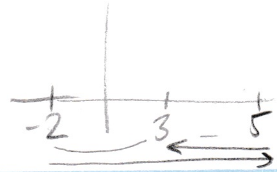
Given $\int_{-2}^{5} f(x)\,dx = 3$ $\quad \int_{-2}^{5} g(x)\,dx = -4$ &

$\int_{3}^{5} f(x)\,dx = 7$

Evaluate

a) $\int_{-2}^{5}(2f(x) - 3g(x))\,dx = 2\int_{-2}^{5} f(x)\,dx - 3\int_{-2}^{5} g(x)\,dx$

$= 2(3) - 3(-4) = \boxed{18}$



5.3

b) $\int_{-2}^{3} f(x)dx = \int_{-2}^{5} f(x)dx - \int_{3}^{5} f(x)dx$

$= 3 - 7$

$= \boxed{-4}$

U-substitution

2 Options

preferred

① Use x's (must back substitute)

② Use u's (must find u values given x's)

Method ①

$\int_{-4}^{0} (2x+6)^4 dx$

$u = 2x+6$

$du = 2dx$

$\frac{1}{2}du = dx$

$= \frac{1}{2}\int_{x=-4}^{x=0} u^4 \, du$

$= \frac{1}{2}\frac{u^5}{5}\Big|_{x=-4}^{x=0} = \frac{1}{10}u^5\Big|_{x=-4}^{x=0}$

Plug u back in

$= \frac{1}{10}(2x+6)^5\Big|_{-4}^{0}$

$= \frac{1}{10}\left[(2(0)+6)^5 - (2(-4)+6)^5\right]$

$= \frac{1}{10}\left[(6)^5 - (-8+6)^5\right]$

$= \frac{1}{10}\left[7776 - (-2)^5\right]$

$= \frac{1}{10}\left[7776 + 32\right]$

$= \frac{1}{10}(7808) = \frac{3905}{4} \approx \boxed{780.8}$

5.3

Method (2)

$$\int_{-4}^{0} (2x+6)^4\,dx$$

$u = 2x+6$
$du = 2dx$
$\frac{1}{2}du = dx$

$= \frac{1}{2}\int_{-2}^{6} u^4\,du$

If $x=0$ $u = 2(0)+6$
$u=6$

If $x=-4$ $u = 2(-4)+6$
$u=-2$

$= \frac{1}{2}\left.\frac{u^5}{5}\right|_{-2}^{6}$

$= \frac{1}{10}\left.u^5\right|_{-2}^{6}$

$= \frac{1}{10}\left[6^5 - (-2)^5\right]$

$= \frac{1}{10}(7776 - (-32))$

$= \frac{1}{10}(7808) = \frac{3904}{5} \approx \boxed{780.8}$

Method 2 is preferred since you do not have to back substitute, but either method will work.

Evals

5.3

ex. $\int_1^2 \frac{x^2}{(x^3+1)^2}\,dx$

Let $u = x^3+1$
$du = 3x^2 dx$
$\frac{1}{3}du = x^2 dx$

$= \frac{1}{3}\int_{x=1}^{x=2} \frac{1}{u^2}\,du$

$X=2 \quad u = 2^3+1 = 9$
$X=1 \quad u = 1^3+1 = 2$

$= \frac{1}{3}\int_2^9 u^{-2}\,du$

$= \frac{1}{3}\left.\frac{u^{-2+1}}{-2+1}\right|_2^9$

$= \frac{1}{3}\left.\frac{u^{-1}}{-1}\right|_2^9$

$= -\frac{1}{3}\cdot\left.\frac{1}{u}\right|_2^9$

$= -\frac{1}{3}\left[\frac{1}{9} - \frac{1}{2}\right]$

$= -\frac{1}{3}\left[\frac{1}{9}\left(\frac{2}{2}\right) - \frac{1}{2}\left(\frac{9}{9}\right)\right]$

$= -\frac{1}{3}\left[\frac{2}{18} - \frac{9}{18}\right] = -\frac{1}{3}\left[\frac{-7}{18}\right] = \boxed{\frac{7}{54}}$

ex. Find the area under the curve $y=f(x)$ over $0 \le x \le 3$ given $y = xe^{-x^2}$

$\int_0^3 xe^{-x^2}\,dx$

Let $u = -x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\int_{x=0}^{x=3} e^u\,du$

If $x=0$, $u=0$
$x=3$, $u = -(3)^2 = -9$

$= -\frac{1}{2}\int_0^{-9} e^u\,du$

$= -\left(-\frac{1}{2}\right)\int_{-9}^{0} e^u\,du = \frac{1}{2}\left.e^u\right|_{-9}^{0} = \frac{1}{2}\left[e^0 - e^{-9}\right] = \boxed{\frac{1}{2}\left(1-\frac{1}{e^9}\right) \approx 0.50}$

More Examples:

$$= \int_{-1}^{0} (-3x^5 - 3x^2 + 2x + 5)\,dx$$

$$= \left(-\cancel{3}\frac{x^6}{2\cancel{6}} - \cancel{3}\frac{x^3}{\cancel{3}} + \frac{2x^2}{2} + 5x\right)\Big|_{-1}^{0}$$

$$= -\frac{1}{2}x^6 - x^3 + x^2 + 5x \Big|_{-1}^{0}$$

$$= 0 - \left(-\frac{1}{2}(-1)^6 - (-1)^3 + (-1)^2 + 5(-1)\right)$$

$$= -\left(-\frac{1}{2}(1) - (-1) + 1 - 5\right)$$

$$= -\left(-\frac{1}{2} + 1 - 4\right)$$

$$= -\left(-\frac{1}{2} - \frac{3}{1}\left(\frac{2}{2}\right)\right)$$

$$= -\left(-\frac{1}{2} - \frac{6}{2}\right) = -\left(-\frac{7}{2}\right) = \boxed{\frac{7}{2}} \approx 3.5$$

ex. $\int^{3} x^3(x+1)\,dx$

$$= \int_{1}^{3} x^4 + x^3\,dx$$

$$= \frac{x^5}{5} + \frac{x^4}{4}\Big|_{1}^{3}$$

$$= \frac{3^5}{5} + \frac{3^4}{4} - \left(\frac{1^5}{5} + \frac{1^4}{4}\right)$$

$$= \frac{243}{5} + \frac{81}{4} - \frac{1}{5} - \frac{1}{4}$$

$$= \boxed{\frac{342}{5}} \approx 68.4$$

$$\int_1^2 \frac{x^3}{(x^4+1)^3}\,dx =$$

Let $u = x^4+1$
$du = 4x^3dx$
$\frac{1}{4}du = x^3dx$

If $x=1$ $u = 1^4+1 = 2$
$x=2$ $u = 2^4+1 = 17$

$$= \frac{1}{4}\int_{x=1}^{x=2} \frac{1}{u^3}\,du$$

$$= \frac{1}{4}\int_2^{17} u^{-3}\,du$$

$$= \frac{1}{4} \left.\frac{u^{-3+1}}{-3+1}\right|_2^{17}$$

$$= \frac{1}{4} \left.\frac{u^{-2}}{-2}\right|_2^{17}$$

$$= -\frac{1}{8}\cdot\left.\frac{1}{u^2}\right|_2^{17}$$

$$= -\frac{1}{8}\left[\frac{1}{17^2} - \frac{1}{2^2}\right]$$

$$= -\frac{1}{8}\left[-\frac{285}{1156}\right] = \boxed{\frac{285}{9248}} = 0.030817$$