Section 5.2 Integration by Substitution

↳ undoes chain rule

$\int x^5 dx = \frac{x^6}{6} + C$

But what if we have

$\int (4x+7)^5 dx = ?$ (Could mult. it out... but Not gonna) too long process

Instead we make a change of variables

Let $u = 4x+7$, then differentiating we get

$\frac{du}{4} = \frac{4dx}{4}$ (Now solve for dx)

$dx = \frac{1}{4} du$ Now we have...

$\int (4x+7)^5 dx = \int u^5 \cdot \frac{1}{4} du$

$= \frac{1}{4} \int u^5 du$

$= \frac{1}{4} \frac{u^6}{6} + C$

$= \frac{1}{24} u^6 + C$ but $u = 4x+7$ Substitute!

$= \boxed{\frac{1}{24}(4x+7)^6 + C}$ ← check by taking deriv. & make sure you get $(4x+7)^5$

Notice: we'd have to use the chain rule to differentiate, so we have to use this process to integrate!

Process

Guessing Game

① Look for an "inner" function $u = u(x)$ that simplifies integrand

② Write integral in terms of only u & du. (Transform all terms with x, ~~and~~ dx into terms with u or du) "No X's floating around"

③ $\int f(x)dx = \int g(u)du$ ← Integrate

④ Substitute $u(x)$ back in

looks composed, think chain rule

this undoes chain rule

ex. a) $\int \sqrt{5x-3}\,dx$ Let $u = 5x-3$

$\frac{du}{5} = \frac{5dx}{5}$

$\frac{1}{5}du = dx$

$= \int \sqrt{u}\,\frac{1}{5}du$

$= \frac{1}{5}\int u^{1/2}\,du$

$= \frac{1}{5}\frac{u^{1/2+1}}{1/2+1} + C$

$= \frac{1}{5}\frac{u^{3/2}}{3/2} + C$

$= \frac{1}{5}\frac{2}{3}u^{3/2} + C$

$= \frac{2}{15}u^{3/2} + C$

$= \boxed{\frac{2}{15}(5x-3)^{3/2} + C}$

ex. $\int x\sqrt{x^2-3}\,dx$

ex. $\int \frac{1}{3x+5}dx$ Let $u = 3x+5$

$\frac{du}{3} = \frac{3dx}{3}$

$\frac{1}{3}du = dx$

$= \frac{1}{3}\int \frac{1}{u}dx$ Integrate

$= \frac{1}{3}\ln|u| + C$ subst. u in

$= \boxed{\frac{1}{3}\ln|3x+5| + C}$

ex. $\int x e^{x^2}dx$ Let $u = x^2$

$\frac{du}{2} = \frac{2x\,dx}{2}$

$\frac{1}{2}du = x\,dx$

$\int e^u\,x\,dx$

$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \boxed{\frac{1}{2}e^{x^2} + C}$

11am

ex. $\int \frac{t+3}{\sqrt{t^2+6t+5}}\,dt$

Let $u = t^2+6t+5$

$du = (2t+6)dt$

$du = 2(t+3)dt$

$\frac{1}{2}du = (t+3)dt$

$= \int \frac{1}{\sqrt{u}}\underbrace{(t+3)dt}_{\frac{1}{2}du}$

$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{2}\,\frac{u^{-1/2+1}}{-1/2+1} + C$

$= \frac{1}{2}\,\frac{u^{1/2}}{1/2} + C$

$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{1}\,u^{1/2} + C$

$= \sqrt{u} + C$

$= \boxed{\sqrt{t^2+6t+5} + C}$

skip if needed

ex. $\int \frac{3x^3}{x^4+7}\,dx$

Let $u = x^4+7$

$du = 4x^3dx$

$\frac{1}{4}du = x^3dx$

$= 3\int \frac{x^3}{x^4+7}\,dx$

$= 3\int \frac{1}{x^4+7}(x^3dx)$

$= \frac{3}{4}\int \frac{1}{u}\,du$

$= \frac{3}{4}\ln|u| + C$

$= \boxed{\frac{3}{4}\ln|x^4+7| + C}$

ex. $\int \frac{x}{x+1}\,dx$ Let $u = x+1 \longrightarrow u-1=x$
$du = dx$
$du = dx$

$= \int \frac{u-1}{u}\,du$

$= \int \left(\frac{u}{u} - \frac{1}{u}\right) du$

$= \int 1 - \frac{1}{u}\,du$

$= u - \ln|u| + C$

$= \boxed{x+1 - \ln|x+1| + C}$

ex. $\int \frac{\ln(5x)}{x}\,dx$ Guess

~~Let $u = 5x$~~
~~$du = 5\,dx$~~

$= \int \ln(5x)\underbrace{\frac{1}{x}dx}_{du}$ Let $u = \ln(5x)$
$du = \frac{1}{\cancel{5}x}\cdot\cancel{5}$
$du = \frac{1}{x}$

$= \int u\,du$

$= \frac{u^2}{2} + C$

$= \boxed{\frac{(\ln(5x))^2}{2} + C}$

$f'(x)$
slope function is given, to find $f(x)$ integrate

ex. $f'(x) = x\sqrt{x^2+5}$ $(2,10)$
$u = x^2+5$ $du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$\int x\sqrt{x^2+5}\,dx$

$\frac{1}{2}\int \sqrt{u}\,du$

$\frac{1}{2}\int u^{1/2}\,du = \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2} = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+5)^{3/2} + C$

In general

$f(x) = \frac{1}{3}(x^2+5)^{3/2} + C$ → we know (2, 10) is an ordered pair.

10   2

$10 = \frac{1}{3}(2^2+5)^{3/2} + C$ → solve for C

$10 = \frac{1}{3}(4+5)^{3/2} + C$

$10 = \frac{1}{3}(9)^{3/2} + C$

$10 = \frac{1}{3}(\sqrt{9})^3 + C$

$10 = \frac{1}{3}(3)^{3} + C$ → $10 = 9 + C$

$-9 \quad -9$

$C = 1$

$\boxed{f(x) = \frac{1}{3}(x^2+5) + 1}$

More Integrals

ex. $\int \sin(5x)\,dx$

$u = 5x$
$du = 5dx$
$\frac{du}{5} = dx \quad \frac{1}{5}du = dx$

$= \frac{1}{5}\int (\sin u)\,du$

$= \frac{1}{5}(-\cos(u)) + C$

$= \boxed{-\frac{1}{5}\cos(5x) + C}$

ex. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ $= 2e^{\sqrt{x}} + C$

$\int x^3 e^{x^4+2}\,dx$

$\int (2x+5)(x^2+5x-2)^7\,dx$ $= \frac{(x^2+5x-2)^8}{8} + C$

ex. $\int e^{6x-5}\,dx$

Two methods

u-substitution

$\int e^{6x-5}\,dx$

Let $u = 6x-5$
$du = 6dx$
$\frac{1}{6}du = dx$

$\frac{1}{6}\int e^{u}\,du$

$\frac{1}{6}e^{u} + C$

$\boxed{\frac{1}{6}e^{6x-5} + C}$

Same Answer!

Algebra 1st & regular integral

$\int e^{6x-5}\,dx$

$= \int e^{6x}\cdot e^{-5}\,dx$ constant

$= e^{-5}\int e^{6x}\,dx$

$= e^{-5}\left(\frac{e^{6x}}{6}\right) + C$

$= \frac{e^{-5}e^{6x}}{6} + C$

$= \boxed{\frac{e^{6x-5}}{6} + C}$

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ex. $\int e^{-x}(3-e^{-x})\,dx$

$u = 3-e^{-x}$
$du = -e^{-x}\,dx$
$-du = e^{-x}\,dx$

$= -\int u\,du$

$= -\frac{u^2}{2} + C$

$= \boxed{-\frac{1}{2}(3-e^{-x})^2 + C}$

Application

55. Revenue. The marginal revenue from the sale of x units of a commodity is estimated to be

$$R'(x) = 50 + 3.5xe^{-0.01x^2} \text{ dollars/unit}$$

where R(x) is revenue in dollars.

a) Find R(x), assuming R(0) = 0

$$R(x) = \int R'(x)\,dx = \int 50 + 3.5xe^{-0.01x^2}\,dx$$

$$= \int 50\,dx + 3.5\int \boxed{x}e^{-0.01x^2}\boxed{dx}$$

$$= 50x + 3.5\int e^{-0.01x^2}\boxed{x\,dx}$$

Let $u = -0.01x^2$

$du = -0.02x\,dx$

$\frac{1}{-0.002}du = \boxed{x\,dx}$

$$= 50x + \frac{3.5}{-0.02}\int e^u\,du$$

$$= 50x + \frac{3.5}{-0.02}e^u + C$$

This is R(x) in general

$$= 50x - 175e^{-0.01x^2} + C$$

Now plug in (0,0) to find C

$$0 = 50(0) - 175e^{-0.01(0)^2} + C$$

$$0 = -175 + C$$

$$C = 175$$

$$\boxed{R(x) = 50x - 175e^{-0.01x^2} - 175}$$

50175

$$R(1{,}000) = 50(1000) - 175e^{-0.01(1000)^2} - 175$$

$$\approx 50175$$

The revenue would be about 50,175 dollars.