Section 5.1 Antidifferentiation

Motivation: How can the rate at which a population is changing be used to predict future population levels?
(if f' is known, how do we find f?)

<u>Antidifferentiation</u> (or the indefinite integration)
A function $F(x)$ is the <u>antiderivative</u> of $f(x)$ if

$$F'(x) = f(x) \quad \text{for all } x \text{ in the domain of } f.$$

or

$$\frac{d}{dx}F = f(x)$$

Let's start by verifying a proposed antideriv.

ex. verify that $F(x) = \frac{1}{3}x^3 + 5x + 2$ is <u>an</u> antideriv. of $f(x) = x^2 + 5$ (i.e. show that $F'(x) = f(x)$)

more than one antideriv. exists.

$$F'(x) = \frac{1}{3}(3x^2) + 5$$
$$= x^2 + 5 = f(x) \checkmark$$

ex. $F(x) = x^3$ is <u>one</u> antideriv. of $f(x) = 3x^2$, so if

$$F(x) = x^3 + 5, \quad F(x) = x^3 - \pi \ldots$$

Family of curves that differ by a constant.

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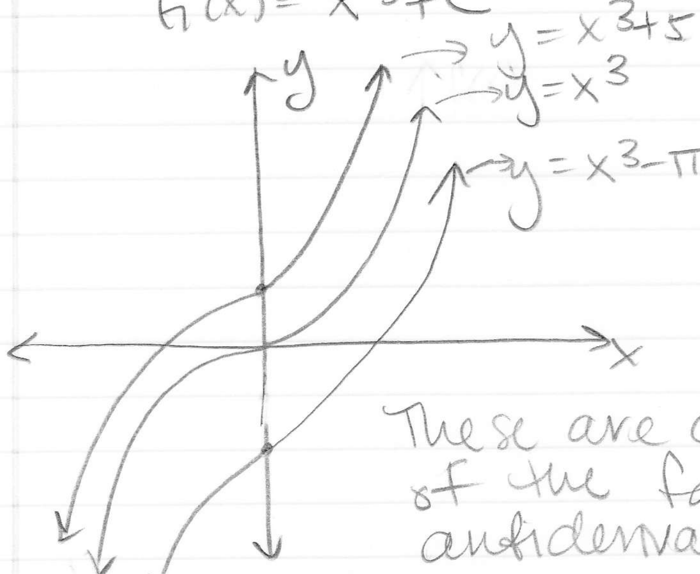
Fundamental Property of Antideriv.

If $F(x)$ is an antideriv. of $f(x)$,
then so is $G(x) = F(x) + C$ for C a constant
(family of curves)

last example
$G(x) = x^3 + C$



These are called members of the family of antiderivatives.

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ex) Find the antiderivatives of

a) $f(x) = -3$
$F(x) = -3x + C$

b) $f(x) = 4x$
$F(x) = 2x^2 + C$

There are rules!! :)

c) $f(x) = e^x$
$F(x) = e^x$

d) $f(x) = 5x^2$
$F(x) = \frac{5}{3}x^3$

Start w/ exponent

The Indefinite Integral

$\int f(x)\,dx = F(x) + C$

antiderivative

Constant of Integrat.

variable of integration (wrt x)

Integral symbol

Integrand

Since $f(x)$ is the derivative of $F(x)$ we could also write

$$\int F'(x)\,dx = F(x) + C$$

(i.e, if you take the derivative of $F(x)+C$, you should get the integrand $F'(x)$)

Rules for Integration

const. $\int k\,dx = kx + C$ for a constant k.

power $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$

log $\int \frac{1}{x}\,dx = \ln|x| + C$ for all $x \neq 0$

exp $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C \qquad k \neq 0$

trig $\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

or $\int \sin(kx)\,dx = -\frac{1}{k}\cos x + C$

$\int \cos(kx)\,dx = \frac{1}{k}\sin x + C$

Verify Power rule / log rule ← if time by taking deriv.

Power rule
$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{\cancel{n+1}}\cancel{(n+1)}\,x^{n+1-1}$$
$$= x^{n+1-1} = x^n \quad \text{so} \quad \boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c}$$

Log rule
If $x>0$, then $|x| = x$
$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x} \checkmark$$

If $x<0$, then $|x| = -x$
$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot -1 = \frac{1}{x} \checkmark$$

So, if $x \neq 0$, $\frac{d}{dx}\ln|x| = \frac{1}{x}$ & $\boxed{\int \frac{1}{x}\,dx = \ln|x| + c}$

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ex) Find the following integrals

a) $\int 5\,dx = 5x + C$

b) $\int x^{21}\,dx = \frac{x^{21+1}}{21+1} + C = \frac{x^{22}}{22} + C$ check: $\frac{1}{\cancel{22}}\cancel{22}x^{21} + 0$ x^{21}

c) $\int \frac{1}{\sqrt{x}}\,dx = \int \frac{1}{x^{1/2}}\,dx = \int x^{-1/2}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$
$$= \frac{x^{-\frac{1}{2}+\frac{2}{2}}}{-\frac{1}{2}+\frac{2}{2}} + c = \frac{x^{1/2}}{\frac{1}{2}} = 2x^{1/2} + c$$
$$= \boxed{2\sqrt{x} + c}$$

d) $\int e^{-6x}\,dx = \boxed{\frac{-1}{6}e^{-6x} + C}$

e) $\int x^{-0.3}\,dx = \frac{x^{-0.3+1}}{-0.3+1} + C = \boxed{\frac{x^{0.7}}{0.7} + C}$

Algebraic Rules

$$\int kf(x)\,dx = k\int f(x)\,dx \quad \text{for constant } k$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

ex) Integrate

a) $\int 5x^9\,dx = 5\int x^9\,dx = 5\,\frac{x^{10}}{10}$

$= \boxed{\frac{1}{2}x^{10} + C}$

b) $\int \frac{3}{\sqrt[3]{x}}\,dx = 3\int \frac{1}{x^{1/3}}\,dx$

$= 3\int x^{-1/3}\,dx$

$= 3\,\frac{x^{-\frac{1}{3}+\frac{3}{3}}}{-\frac{1}{3}+\frac{3}{3}} + C$

$= 3\,\frac{x^{2/3}}{2/3} + C$

$= 3\left(\frac{3}{2}\right)x^{2/3} + C$

$= \boxed{\frac{9}{2}x^{2/3} + C}$

or $= \boxed{\frac{9}{2}\left(\sqrt[3]{x}\right)^2 + C}$

c) $\int (3x^4 - 5x^2 + 1)\,dx$

$= \int 3x^4\,dx - \int 5x^2\,dx + \int 1\,dx$

$= 3\int x^4\,dx - 5\int x^2\,dx + \int 1\,dx$

$= 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + 1 \cdot x + C$

$= \boxed{\frac{3}{5}x^5 - \frac{5}{3}x^3 + x + C}$

d) $\int \left(\frac{1}{x^2} - \frac{1}{x}\right)dx$

$= \int x^{-2}\,dx - \int x^{-1}\,dx$

$= \frac{x^{-2+1}}{-2+1} - \ln|x| + C$

$= \frac{x^{-1}}{-1} - \ln|x| + C$

$= \boxed{-\frac{1}{x} - \ln|x| + C}$

Check by taking deriv. & see if you get what you started with!

More Examples
Integrate

a) $\int (e^t+1)^2\,dt$ ← No rules for this yet! (use algebra)

$= \int (e^t+1)(e^t+1)\,dt$ ← FOIL

$= \int (e^{t+t}+e^t+e^t+1)\,dt$

$= \int (e^{2t}+2e^t+1)\,dt$

$= \frac{1}{2}e^{2t}+2e^t+t+C$

b) $\int (x^3-2x^2)(\frac{1}{x}-5)\,dx$

$= \int \left(\frac{x^3}{x}-5x^3-\frac{2x^2}{x}+10x^2\right)dx$

$= \int (x^2-5x^3-2x+10x^2)\,dx$

$= \int (-5x^3+11x^2-2x)\,dx$

$= -5\frac{x^4}{4}+11\frac{x^3}{3}-\frac{2x^2}{2}+C$

$= \boxed{-\frac{5}{4}x^4+\frac{11}{3}x^3-x^2+C}$

c) $\int \frac{x^2+2x+1}{x^2}\,dx$ Break down 3 fractions

$= \int \left(\frac{x^2}{x^2}+\frac{2x}{x^2}+\frac{1}{x^2}\right)dx$

$= \int \left(1+\frac{2}{x}+\frac{1}{x^2}\right)dx$

$= \int \left(1+2\frac{1}{x}+x^{-2}\right)dx$

$= x+2\ln|x|+\frac{x^{-2+1}}{-2+1}+C$

$= x+2\ln|x|+\frac{x^{-1}}{-1}+C$

$= \boxed{x+2\ln|x|-\frac{1}{x}+C}$

differential equation w/ given initial value

Solve the given initial value problem for $y=f(x)$ given
$\frac{dy}{dx} = 3x-2$ where $y=2$ when $x=-1$

$$y=\int \frac{dy}{dx}dx = \int 3x-2\,dx = 3\frac{x^2}{2} - 2x + c$$

$$y = \frac{3}{2}x^2 - 2x + c$$

$x=-1$
$y=2$

$$2 = \frac{3}{2}(-1)^2 - 2(-1) + c$$

$$2 = \frac{3}{2} + 2 + c$$
$$-2 \qquad\qquad -2$$

$$0 = \frac{3}{2} + c$$
$$-\frac{3}{2} \qquad -\frac{3}{2}$$

$$-\frac{3}{2} = c$$

So, $\boxed{y = \frac{3}{2}x^2 - 2x - \frac{3}{2}}$ is the exact function to the initial value problems.

ex. The slope $f'(x)$ is given for $y=f(x)$ as well as (a,b) on curve $f(x)$. Use this info to find $f(x)$.
$f'(x) = e^{-x} + x^2$ $\quad (0,4)$

$$y=f(x) = \int f'(x)dx = \int e^{-x} + x^2 dx = -1e^{-x} + \frac{x^3}{3} + c$$

$$y = -e^{-x} + \frac{1}{3}x^3 + c$$

$$4 = -e^{-0} + \frac{1}{3}(0)^3 + c$$
$$4 = -e^0 + 0 + c$$
$$4 = -1 + c$$
$$+1 \qquad +1$$
$$c = 5$$

$$\boxed{f(x) = -e^{-x} + \frac{1}{3}x^3 + 5}$$