where does e come from?

Section 4.3 Differentiation of Exponential & Logarithmic Functions

Applications of Exp/logs
- Pop. growth (r.o.c of pop)
- Compound interest
- Concent. of drug (r.o.c of drug)
- Carbon Dating / Radioactive decay

The Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

(proof in book pg 325)

ex) Differentiate

a) $f(x) = 5 \cdot e^x + 2x^2 + 3$

$\boxed{f'(x) = 5 \cdot e^x + 4x}$

b) $g(x) = x^2 e^x$

$g'(x) = x^2 e^x + e^x 2x$

$\boxed{g'(x) = xe^x(x+2)}$

c) $f(x) = \dfrac{x^3}{e^x+2}$

$f'(x) = \dfrac{(e^x+2)\cdot 3x^2 - x^3(e^x)}{(e^x+2)^2}$

$\boxed{f'(x) = \dfrac{3x^2e^x + 6x^2 - x^3e^x}{(e^x+2)^2}}$

The Chain Rule

$$\frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot u'(x) \quad \text{or} \quad \left(e^{(\text{stuff})}\right)' = e^{(\text{stuff})}(\text{stuff})'$$

ex. $f(x) = e^{x^2}$ $\quad f'(x) = e^{x^2} \cdot (x^2)' = e^{x^2} \cdot 2x = \boxed{2xe^{x^2} = f'(x)}$

ex. $f(x) = \dfrac{e^{-3x}}{x^2+1}$

$f'(x) = \dfrac{(x^2+1)\,e^{-3x}\cdot(-3) - e^{-3x}(2x)}{(x^2+1)^2}$

$= \dfrac{e^{-3x}\left[-3(x^2+1) - 2x\right]}{(x^2+1)^2}$

$= \dfrac{e^{-3x}\left[-3x^2-3-2x\right]}{(x^2+1)^2}$

$= \dfrac{e^{-3x}(-3x^2-2x-3)}{(x^2+1)^2}$

$\boxed{f'(x) = \dfrac{-e^{-3x}(3x^2+2x-3)}{(x^2+1)^2}}$

The Derivative of $\ln x$ (for all $x>0$)

$\dfrac{d}{dx}\ln x = \dfrac{1}{x}$

ex. Differentiate

a) $f(x) = 2x^2+6x-\ln x$

$\boxed{f'(x) = 4x+6-\dfrac{1}{x}}$

b) $f(x) = x\ln x$

$f'(x) = x\,\dfrac{1}{x} + \ln x\cdot 1$

$\boxed{f'(x) = 1+\ln x}$

Algebra 1st!

c) $f(x) = \dfrac{\ln(x^2)}{x^4} = \dfrac{2\ln x}{x^4}$

$f'(x) = \dfrac{x^4\cdot 2\cdot\frac{1}{x} - 2\ln x\cdot 4x^3}{x^8}$

$= \dfrac{2x^3 - 2\ln x\cdot 4x^3}{x^8} = \dfrac{2x^3(1-4\ln x)}{x^8} = \boxed{\dfrac{2(1-4\ln x)}{x^5}}$

ex. $g(t) = (t + \ln t)^{3/2}$

$g'(t) = \frac{3}{2}(t + \ln t)^{\frac{3}{2}-\frac{2}{2}} \cdot (t + \ln t)'$

$$\boxed{g'(t) = \frac{3}{2}(t + \ln t)^{\frac{1}{2}} \cdot \left(1 + \frac{1}{t}\right)}$$

Chain Rule for $\ln(u(x))$ $\quad u(x) > 0$

$$\frac{d}{dx} \ln(u(x)) = \frac{1}{u(x)} \cdot u'(x) = \frac{u'(x)}{u(x)}$$

or

$$(\ln(\text{stuff}))' = \frac{1}{\text{stuff}} (\text{stuff})' = \frac{(\text{stuff})'}{\text{stuff}} \qquad \text{stuff} > 0$$

ex. $f(x) = \ln(2x^3 + 8x)$

$f'(x) = \frac{1}{2x^3+8x}(2x^3+8x)'$

$= \frac{1}{2x^3+1}(6x+8)$

$$\boxed{f'(x) = \frac{6x+8}{2x^3+1}}$$

ex. $f(x) = x \cdot \ln(x^3)$

$f'(x) = x \cdot \frac{1}{x^3} \cdot 3x^2 + \ln(x^3)(1)$

$= \frac{3x^{\cancel{3}}}{\cancel{x^3}} + \ln(x^3)$

$$\boxed{f'(x) = 3 + \ln x^3}$$

Derivative of b^x & $\log_b x$ $\quad b>0,\ b\neq 1$

$$\frac{d}{dx} b^x = \ln(b)\cdot b^x \quad \text{for all } x$$

$$\frac{d}{dx} \log_b x = \frac{1}{x\ln b} \quad \text{for all } x>0$$

ex. Differentiate

a) $f(x) = 2^x$

$$\boxed{f'(x) = \ln(2)\cdot 2^x}$$

b) $f(x) = 3^x(6x+2)$

$f'(x) = 3^x(6) + (6x+2)\ln(3)3^x$

$$\boxed{f'(x) = 3^x[6 + \ln 3(6x+2)]}$$

c) $g(x) = \log_4 x$

$$\boxed{g'(x) = \frac{1}{x\cdot\ln 4}}$$

d) $h(t) = \dfrac{t^4+2t}{\log_{10} t}$

$$\boxed{h'(t) = \frac{3\log_{10} t - (t^4+2t)\left(\frac{1}{t\ln 10}\right)}{(\log_{10} t)^2}}$$

Chain Rule for $b^{u(x)}$ & $\log_b(u(x))$

$$\frac{d}{dx} b^{u(x)} = \ln b\cdot b^{u(x)}\cdot u'(x)$$

&

$$\frac{d}{dx}\log_b(u(x)) = \frac{1}{u(x)\cdot\ln b}\cdot u'(x) = \frac{u'(x)}{u(x)\cdot\ln b}$$

Only 4.3 due

ex a) $f(x) = 5^{x^2+6x+3}$

$\boxed{f'(x) = \ln 5 \cdot 5^{x^2+6x+3} \cdot (2x+6)}$

b) $g(x) = (4x^2+15) \cdot 20^{3x^2+5x}$

$\boxed{g'(x) = (4x^2+15) \ln 20 \cdot 2^{3x^2+5x}(6x+5) + 20^{3x^2+5x}(8x)}$

c) $f(x) = \log_7 (x^2 + \frac{1}{x}) = \log_7 (x^2 + x^{-1})$

$\boxed{f'(x) = \frac{(2x - 1x^{-2})}{(x^2+x^{-1})\ln 7}}$

d) $g(x) = (x^2 + \log_7 x)^4$

$\boxed{g'(x) = 4(x^2 + \log_7 x)^3 \left(2x + \frac{1}{x \ln 7}\right)}$

e) $f(x) = e^{2x} \cdot 5^x$

$= e^{2x} \ln 5 \cdot 5^x + 5^x e^{2x} \cdot 2$

$\boxed{f'(x) = 5^x e^{2x} (\ln 5 + 2)}$

f) $f(x) = 6^{2x} \log_9 x^2$

$\boxed{f'(x) = 6^{2x} \frac{2x}{x^2 \cdot \ln 9} + \log_9 x^2 \cdot \ln 6 \cdot 6^{2x} \cdot 2}$

Find the equation of the tangent line to $f(x) = e^x(x+1)$ at $x=0$

$f(0) = e^0(0+1) = 1(1) = 1$

$(0, 1)$

$f'(x) = e^x(1) + (x+1)e^x$

$= e^x(1 + x + 1)$

$f'(x) = e^x(x+2)$

$f'(0) = e^0(0+2)$

$= 1(2)$

$m = 2$

$\boxed{y = 2x + 1}$

1st*

ex) Find the absolute max/min of

$$g(x) = \frac{e^x}{2x+1} \quad \text{for} \quad 0 \le x \le 1$$

$$g(0) = \frac{e^0}{2(0)+1} = \frac{1}{1} = 1$$

$$g(1) = \frac{e^1}{2(1)+1} = \frac{e}{3} \approx 0.9061$$

$$g'(x) = \frac{(2x+1)e^x - e^x(2)}{(2x+1)^2}$$

$$g'(x) = \frac{e^x(2x+1-2)}{(2x+1)^2}$$

$$g'(x) = \frac{e^x(2x-1)}{(2x+1)^2}$$

$$0 = e^x(2x-1)$$

$\cancel{e^x = 0}$ $\quad 2x-1=0$

$$2x = 1$$

$$x = \frac{1}{2}$$

$$g(\tfrac{1}{2}) = \frac{e^{1/2}}{2(\frac{1}{2})+1} = \frac{e^{1/2}}{1+1} = \frac{e^{1/2}}{2} \approx 0.8244$$

E.P.'s

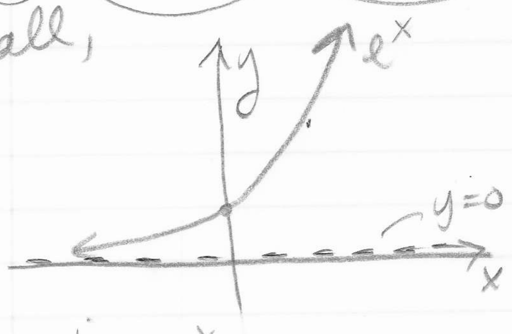
(0, 1)
(1, 0.9061)

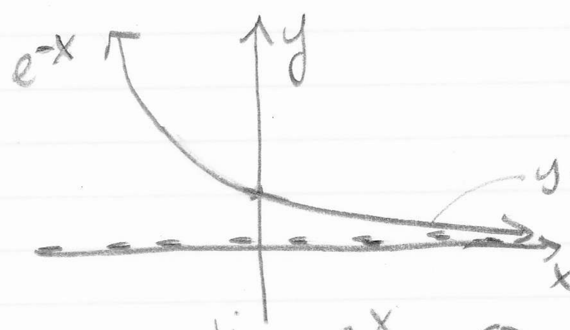
C.P.'s

$(\frac{1}{2}, 0.8244)$

$\boxed{\text{There is an absolute max @ } (0,1) \text{ \& min @ } (\tfrac{1}{2}, \tfrac{e^{1/2}}{2}) \text{ or } (\tfrac{1}{2}, 0.8244).}$

Recall,





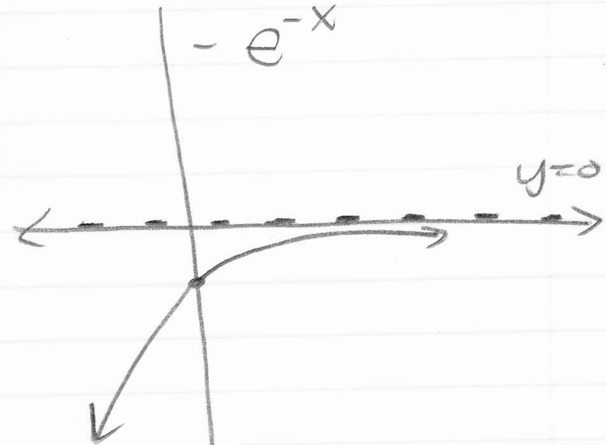
$$\lim_{x\to\infty} e^x = \infty \qquad\qquad \lim_{x\to\infty} e^x = 0$$

$e^x \neq 0$ Never equal to zero
$e^{-x} \neq 0$ Always Positive!

$-e^x$

$\lim_{x\to\infty} -e^x = -\infty$

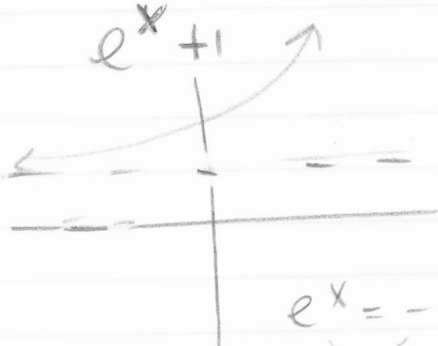
$-e^{-x}$



$\lim_{x\to\infty} -e^{-x} = 0$

Always negative
Never = zero
$-e^x \neq 0$
$-e^{-x} \neq 0$

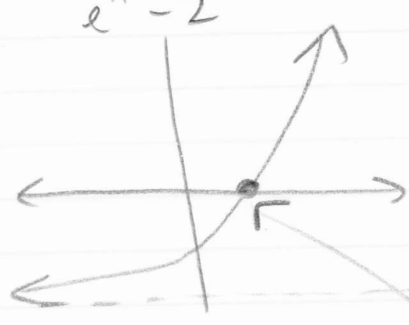
$e^x + 1$



$e^x = -1$

Never equal to a negative

$e^x - 2$



$e^x - 2 = 0$

$e^x = 2$

$\ln e^x = \ln 2$

$\boxed{x = \ln 2}$

$$\lim_{t\to\infty} \frac{21}{1+25e^{-0.3t}}$$

$$= \frac{\lim_{t\to\infty} 21}{\lim_{t\to\infty} 1 + 25 \lim_{t\to\infty} e^{-0.3t} \to 0} = \frac{21}{1+25(0)}$$

$$= \frac{21}{1} = \boxed{21}$$

Math 105 - Section 4.3 - Lecture Notes

1. The population (in millions) of a country is modeled by the function
   $P(t) = 100e^{-0.1t}$
   a. What was the initial population?

   b. At what percentage rate is the population changing with respect to time?

   c. What is the rate of change of population after 10 years? Is it increasing or decreasing?

   d. What happens in the long run? (i.e. as t approaches infinity)

2. Suppose the percentage of alcohol in the blood t hours after consumption is given by $C(t) = 0.12te^{-t/2}$
   a. At what rate is the blood alcohol changing at time t?
   b. How much time passes before the blood alcohol level begins to decrease?
   c. At what rate is the blood alcohol level changing after 4 hours?

Handout!

ex. The population of a country is modelled by $P(t) = 100e^{-0.1t}$ P million (t yrs)

a) What was the initial pop? i.e. $t=0$

$P(0) = 100e^{0}$

$= 100(1) = 100$

$\boxed{\text{The population was 100 million.}}$

b) What is the r.o.c of pop. after 10 yrs? Pop increasing/decreasing?

$P'(t) = 100e^{-0.1t}(-0.1)$

$= -100(0.1)e^{-0.1t}$

$P'(t) = -10e^{-0.1t}$

$P'(10) = -10e^{-0.1(10)}$

$= -10e^{-1}$

$= -\frac{10}{e} \approx$

$\boxed{\text{The pop is decreasing at a rate of } 10/e \text{ million people per year.}}$

c) What happens in the long run? i.e. when $t \to \infty$.

$P(t) = 100e^{-0.1t}$

$= \frac{100}{e^{0.1t}} = \frac{100}{e^{\infty}} = \frac{\#}{\infty} = \boxed{0}$

$\boxed{\text{In the long run the pop is approaching zero.}}$

d) At what percentage is the pop. changing w.r.t. constant r.o.c

$P = P_0 e^{rt}$ r = rate as a decimal!

$-0.1 \Rightarrow -10\%$ $\boxed{\text{Decreasing at a rate of } 10\%}$

#2 on Worksheet/Handout

a) $C(t) = 0.12\,t\,e^{-t/2}$ product rule

$$C'(t) = 0.12t\left(e^{-t/2}\cdot -\tfrac{1}{2}\right) + e^{-t/2}(0.12)$$

$$\boxed{C'(t) = e^{-t/2}\left(-0.06t + 0.12\right)}$$

b) When is $C'(t)$ negative?

$e^{-t/2}$ is always positive

So we need to know when

$$-0.06t + 0.12 < 0$$

$$-0.06t < -0.12$$

$$t > \frac{-0.12}{-0.06}$$

$$t > 2$$

After two hours the concentration starts to decrease.

c) After 4 hours $\Rightarrow$ Find $C(4)$

$$c'(4) = e^{-4/2}\left[-0.06(4) + 0.12\right]$$

$$e^{-2}\left[-0.24 + 0.12\right]$$

$$e^{-2}(-0.12) = -0.016$$

The concentration is decreasing at a rate of 0.02% per hour.