Section 4.2 Logarithmic Functions

Logarithmic Function \( x > 0 \), \( b > 0 \), \( b \neq 1 \)

\[ y = \log_b x \text{ if and only if } b^y = x \]

\( \rightarrow \) the exponent we raise \( b \) to get \( x \) out.

\( \left( \text{logs & exponential are inverses} \right) \)

\[ \log_b x = y \] \( \Rightarrow \) \( b^y = x \)

\( \log_b b^x = x \)

\( a) \) \( \log_2 16 = 4 \) \( \text{ since } 2^4 = 16 \)

\( b) \) \( \log_{10} 1000 = 3 \) \( \text{ since } 10^3 = 1000 \)

\( c) \) \( \log_5 \frac{1}{125} = -3 \) \( \text{ since } 5^{-3} = \frac{1}{125} \)

**Example: Solve for \( x \)**

\( \text{Standard Base: } \log_{10} x \)

Assume base 10, like # System (Calc).

\( a) \) \( \log_2 x = -3 \)

\[ 2^{-3} = x \]

\[ \frac{1}{8} = x \]

\( b) \) \( \log_3 243 = x \)

\[ 3^x = 243 \]

\[ 3^5 = 243 \]

\[ x = 5 \]

\( c) \) \( \log_9 27 = \frac{3}{2} \)

\[ \left( x^3 \right)^{\frac{1}{2}} = 27 \]

\[ x = 27 \]

\[ \frac{3^{11}}{3^3} = 3^8 \]

\[ x = 3 \]

**Rules of Logarithms**

\( b > 0, b \neq 1 \) \( \text{ then } \)

\[ \log_b 1 = 0 \]

\[ \log_b b = 1 \]

and if \( u \) and \( v \) are positive \( \#s \), we have

\( \rightarrow \) \( \log_b u = \log_b v \text{ if and only if } u = v \)

\( \rightarrow \) \( \log_b(uv) = \log_b u + \log_b v \)

\( \rightarrow \) \( \log_b \left( \frac{u}{v} \right) = \log_b u - \log_b v \)

\( \rightarrow \) \( \log_b b^u = u \)

\( \rightarrow \) \( \log_b u^r = r \log_b u \)
Use logs to rewrite in terms of $\log_3 2$ and $\log_3 5$.

\[
\begin{align*}
\log_3 200 &= \log_3 (2^3 \cdot 5^2) \\
&= \log_3 2^3 + \log_3 5^2 \\
&= 3 \log_3 2 + 2 \log_3 5
\end{align*}
\]

Use properties of logs to simplify (expand using prop's)

a) $\log_5 (2^6 y^4)$

\[
= \log_5 2^6 + \log_5 y^4
= 6 \log_5 2 + 4 \log_5 y
\]

b) $\log_6 \left( \frac{x^5 \sqrt{y}}{2^3} \right)$

\[
= \log_6 x^5 + \frac{1}{2} \log_6 y - 3 \log_6 2
\]
Graphs of logarithmic functions
Exponentials and logs are inverses (Inverses are symmetric w/r/t the line $y = x$)

To find inverse, switch $x$ and $y$

$y = a^x$

Domain $(0, \infty)$
Range $(-\infty, \infty)$

$y = \log_b x$

$\rightarrow$ Continuous for all $x > 0$
$\rightarrow$ $x = 0$ is a V.A.
$\rightarrow$ The $x$ int $(1,0)$, no $y$ int
$\rightarrow$ For $x > 0$, the graph is increasing when $b > 1$, and $0 < b < 1$ the graph is decreasing.

The Natural Logarithm
Most common (only has one base)
$\log_e x$ and is denoted $\ln x$ "el en of $x$"

For $x > 0$, $y = \ln x$ if and only if $e^y = x$

Graph $y = \ln x$
$e^x$ and $\ln x$ are inverses

That is

$$e^{\ln x} = x \quad x > 0$$

$\ln (e^x) = x$ for all $x$ (same rules as logs - $\ln$ is just a "special log")

ex. Break down using props of logs

a) $\ln \sqrt[3]{x^2-x}$

$$= \ln (x^2-x)^{1/3}$$

$$= \ln (x(x-1))^{1/3}$$

$$= \frac{1}{3} \left( \ln (x(x-1)) \right)$$

$$= \frac{1}{3} (\ln x + \ln (x-1))$$

ex. $\ln \sqrt[3]{x+1}$

$$= \ln (x+1)^{1/3} x^{-2} (x^2-1)^{-1/2}$$

$$= \frac{1}{3} \ln (x+1) - 2 \ln x - \frac{1}{2} \ln (x^2-1)$$

Recall $b^x = b^y$ if and only if $x = y$

Also if $x = y$, then $\log_b x = \log_b y$ (take log of both sides)

and if $x = y$, then $\ln x = \ln y$

ex. Solve $e^{5x} = 4$

$$\ln e^{5x} = \ln 4$$

$$5x = \ln 4$$

$$x = \frac{\ln 4}{5}$$

exact

You could take log of both sides but calc is easier! $x \ln 2 = \ln 5$
What about finding an approx. of 
\( \log_2 5 \)?

To put in calc we have to 
use a formula

**Change of Base Formula**

\[
\log_b a = \frac{\log_c a}{\log_c b} \\
\text{or} \quad = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}
\]

\[
\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} \quad \text{or} \quad = \frac{\ln 5}{\ln 2}
\]

Try both on calc!

**ex) Solve for x**

a) \( \log_5 (3x+2) = 1 \)

Just convert to exponential!

\[
5^1 = 3x + 2 \\
-2 \quad -2
\]

\[
3 = 3x \\
x = 1
\]

Check it 
\[
5(1) + 2 = 3 + 2 = 5 > 0
\]

b) \( \ln 7 = 10 + 6e^{2x} \\
-10 \quad -10
\]

\[
\frac{7}{6} = \frac{6e^{2x}}{6} \\
\frac{7}{6} = e^{2x} \\
\ln \frac{7}{6} = \ln e^{2x}
\]

\[
\frac{1}{2} \ln \frac{7}{6} = x \\
\approx 0.077075
\]
c) \( \ln x = 2 (\ln 3 - \ln 5) \)
\[ \ln x = 2 \ln \frac{3}{5} \]
\[ \ln x = \ln \left(\frac{3}{5}\right)^2 \]
\[ e^{\ln x} = e^{\ln \left(\frac{3}{5}\right)^2} \]
\[ x = \left(\frac{3}{5}\right)^2 \]
\[ x = \frac{9}{25} \]

d) \(-3\ln x = a\)
\[ \frac{-3}{-3} \ln x = \frac{-a}{-3} \]
\[ \ln x = \frac{a}{3} \]
\[ e^{\frac{a}{3}} = x \]