Section 3.5 Applied Optimization

<u>Strategy</u>

① Come up with Ⓐ function in two variables that you wish to max/minimize. (Declare variables/draw pic)

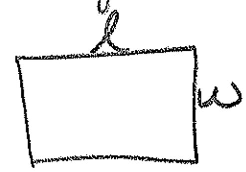
② Come up with an Ⓑ <u>equation</u>/<u>inequality</u> relating the two variables (Translate math → english)

③ Solve equation Ⓑ for one variable and substitute into function Ⓐ so that it is a function of only one variable.

④ Since function f is to be optimized, find f' & all c.v.'s of f. Then find the max/min using methods in 3.4. (2nd deriv. test for max/min). Remember, you may have to check the value of $f(x)$ at endpoints of an interval.

⑤ Interpret your results & answer in a sentence

ex. A gardener has 60ft of fencing and is building a garden. a) Find the <u>dimension</u> that would give the garden of the largest area. b) What is the max area?



a) Let l = length w = width

Ⓐ Area $= l \cdot w$

Ⓑ Perimeter $= 60$

$$2l + 2w = 60 \quad \text{solve for } l$$

$$\frac{2l}{2} = \frac{60-2w}{2} \quad \text{substitute into Ⓐ}$$

$$l = \boxed{30 - w}$$

$$A(w) = (30-w)(w)$$
$$= 30w - w^2$$
$$A(w) = -w^2 + 30w$$ Now find maximum, find $A'(w) = 0$

$A(w) = -w^2 + 30w$

$A'(w) = -2w + 30$

$0 = -2w + 30$

$2w = +30$

$w = 15$ possible max/min

Test using $A''(w) = -2$ ← constant func.

$A''(15) = -2 < 0$ ← max @ $w = 15$

(Think about $A(w)$ being a parabola opens ↓)

$w = 15$ $\quad l = 30 - w$

$= 30 - 15$

$l = 15$

$\boxed{\text{The length and width are both 15ft.}}$

b) Max area

① $\text{Area} = l \cdot w$

$= 15 \cdot 15$

$= 225 ft^2$

OR

② $A(w) = -w^2 + 30w$

$= -(15)^2 + 30(15)$

$= -225 + 450$

$= 225 ft^2$

$$\begin{array}{r} 450 \\ -225 \\ \hline 225 \end{array}$$

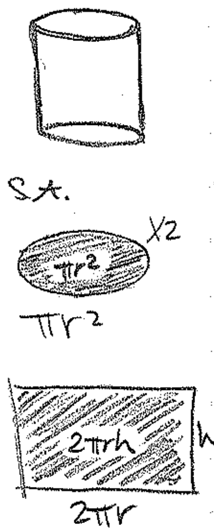
$$\begin{array}{r} 15 \\ \times 15 \\ \hline 75 \\ 150 \\ \hline 225 \end{array}$$

$\boxed{\text{The max area would be } 225ft^2}$

ex. A soup container is to have a $\boxed{16\pi \text{ cm}^3}$ volume volume. The material for the top and bottom costs twice as much per square cm as the side. Find the dimensions of the can with

a) the <u>smallest</u> surface area

b) the cheapest production cost



a) Let h = height & r = radius

function has to do with S.A.

<u>Smallest S.A.</u>

Ⓐ $S = 2\pi r^2 + 2\pi r h$

Ⓑ Has to do with given info/#

Volume $= 16\pi$

$\frac{\pi r^2 h}{\pi r^2} = \frac{16\cancel{\pi}}{\cancel{\pi} r^2}$ solve for h

$h = \boxed{\frac{16}{r^2}}$ $r \neq 0$

$S(r) = 2\pi r^2 + 2\pi \cancel{r}\left(\frac{16}{r^{\cancel{2}}}\right)$

$S(r) = 2\pi r^2 + \frac{32\pi}{r}$ ← Find <u>min</u>

$S(r) = 2\pi r^2 + 32\pi r^{-1}$ $S'(r) = 0$

$S'(r) = 4\pi r - 32\pi r^{-2}$

$S'(r) = 4\pi r - \frac{32\pi}{r^2}$ $r \neq 0$

$0 = 4\pi r - \frac{32\pi}{r^2}$

$\cancel{r^2}\left(\frac{32\pi}{\cancel{r^2}}\right) = (4\pi r)r^2$

$\frac{32\cancel{\pi}}{4\cancel{\pi}} = \frac{4\pi r^3}{4\pi}$

$8 = r^3$

$2 = r$

$\boxed{r = 2}$ C.V. Make sure its a min

min surface area

S.A. $= 2\pi(2)^2 + 2\pi(2)(4)$

$= 8\pi + 16\pi$

$= 24\pi \text{ cm}^2$

$S''(r) = 4\pi + 64\pi r^{-3}$

$S''(2) = 4\pi + \frac{64\pi}{(2)^3} > 0$ ∪ ← min @ r = 2 ✓

$r = 2$ Ⓑ $h = \frac{16}{r^2} = \frac{16}{2^2} = \boxed{4 = h}$

The dimensions that yield minimum S.A. are radius of 2 cm & height of 4 cm

b) Cheapest production cost

let r = radius h = hieght x = cost per cm^2 of material (consant) \$3

A $\text{Cost} = 2x\left(\text{Area of top \& bottom}\right) + x\left(\text{Area of "side"}\right)$ B

$$= 2x(2\pi r^2) + x(2\pi r h) \quad \text{but } h = \boxed{\frac{16}{r^2}}$$

$$= 4x\pi r^2 + 2x\pi \cancel{r}\left(\frac{16}{r^{\cancel{2}}}\right)$$

$$C(r) = 4x\pi r^2 + \frac{32x\pi}{r}$$

$$C(r) = 4x\pi r^2 + 32x\pi r^{-1}$$

$$= 4x\pi(r^2 + 8r^{-1})$$

$$C'(r) = 4x\pi(2r - 8r^{-2})$$

$$\frac{0}{4x\pi} = \frac{4x\pi(2r - 8r^{-2})}{4x\pi}$$

$$0 = 2r - \frac{8}{r^2}$$

$$\cancel{r^2}\left(\frac{8}{\cancel{r^2}}\right) = (2r)r^2 \quad r \neq 0$$

$$\frac{8}{2} = \frac{2r^3}{2}$$

$$4 = r^3$$

$$\boxed{r = \sqrt[3]{4}}$$

Check $C''(r) = 4x\pi(2 + 16r^{-3})$

$$= \frac{4x\pi}{2 + 16(\sqrt[3]{4})}$$

$= +$ concave up

min @ $r = \sqrt[3]{4}$

$$h = \frac{16}{r^2} = \frac{16}{(\sqrt[3]{4})^2}$$

A radius of $\sqrt[3]{4}$ cm and a height of $\frac{16}{(\sqrt[3]{4})^2}$ would minimize production costs.

if extra time

8. A farmer wants to enclosed a rectangular field with an area of $400 m^2$.
a) What is the smallest amount of fencing needed? — minimize fxn.
b) What are the dimensions?

a) Let w - width, l - length

what do we want to max/minimize? A

A Perimeter $= 2l + 2w$

$P = 2l + 2w$

B. Area $= 400$

$l \cdot w = 400$

$l = \boxed{\frac{400}{w}} \xrightarrow{\text{subst. into A}} P(w) = 2l + 2w$

$= 2\left(\frac{400}{w}\right) + 2w$

$P(w) = \frac{800}{w} + 2w \quad w \neq 0$

$= 800w^{-1} + 2w$

Now find c.p.s.

$P'(w) = -800w^{-2} + 2$

$0 = -\frac{800}{w^2} + 2$

$\frac{+800}{w^2} = 2 \quad w \neq 0$

$\frac{800}{2} = \frac{2w^2}{2}$

$400 = w^2$

$w = \pm\sqrt{400}$

$w = \pm 20$

$\boxed{w = 20}$

Make sure its a min

$f''(w) = 1600w^{-3}$

$f''(w) = \frac{1600}{w^3}$

$f''(20) = \frac{1600}{20^3} \quad \frac{+}{+} = +$

← minimum

*Answer question.
Now find smallest perimeter!

$l = \frac{400}{20} = 20$

Perimeter $= 2l + 2w$

$= 2(20) + 2(20)$

$= 40 + 40 = 80$

The farmer would need 80 meters.
with a length and width of 20 meters