Section 3.5 Applied Optimization

Strategy
① Come up with a function that you wish to maximize/minimize. (Declare variables/draw picture)
② Come up with an equation/inequality relating the two variables (translating math language)
③ Solve equation ② for one variable and substitute into function ④ so that it is a function of only one variable
④ Since function ① is to be optimized, find ③ & all c.v.s of ①. Then find the max/min using methods in 3.4.(Epsilon test for max/min).
Remember, you may have to check the value of ①(0) at endpoints of an interval
⑤ Interpret your results and answer in a sentence

8. A gardener has 60 ft of fencing and is building a garden. Find the dimensions that would give the garden of the largest area.
   a) What is the max area?
      ① Let \( l = \) length, \( w = \) width
      ② Area = \( l \times w \)
      ③ Perimeter = 60
         \( 2l + 2w = 60 \) \( \text{Solve for} \ l \)
            \( \frac{2l}{2} = \frac{60 - 2w}{2} \)
            \( l = \frac{30 - w}{2} \)
      ④ Substitute into ②
         \( A(w) = (30 - w) \cdot w \)
            \( = 30w - w^2 \)
      \( A'(w) = -w^2 + 30w \) \( \text{New find maximum, find} \ A'(w) = 0 \)
\[ A(w) = -w^2 + 30w \]
\[ A'(w) = -2w + 30 \]
\[ 0 = -2w + 30 \]
\[ 2w = 30 \]
\[ w = 15 \] possible max/min

Test using \[ A''(w) = -2 \] is constant func.
\[ A''(15) = -2 < 0 \] \( \Rightarrow \) max @ \( w = 15 \)

\[ w = 15 \]
\[ l = 30 - w \]
\[ = 30 - 15 \]
\[ = 15 \]
\[ w = 15 \]

The length and width are both 15 ft.

b) Max area

\[ \text{Area} = l \cdot w \]
\[ = 15 \cdot 15 \]
\[ = 225 \text{ ft}^2 \]

\[ \frac{450}{225} \]
\[ = 2 \text{ ft}^2 \]

The max area would be 225 ft\(^2\).
A soup container is to have a volume of \(16\pi \text{cm}^3\). The material for the top and bottom costs twice as much per square cm as the side. Find the dimensions of the can with:

a) the smallest surface area
b) the cheapest production cost

a) Let \(h = \text{height}\) and \(r = \text{radius}\)

**Smallest Surface Area**

\[
S(r) = 2\pi r^2 + 2\pi r h
\]

\[
S(r) = 2\pi r^2 + \frac{16}{r}
\]

\[
S'(r) = 4\pi r - \frac{32\pi}{r^2}
\]

\[
0 = 4\pi r - \frac{32\pi}{r^2}
\]

\[
4\pi r^3 = 32\pi
\]

\[
r^3 = \frac{32\pi}{4\pi}
\]

\[
r = \sqrt[3]{8} = 2
\]

C.V.: Make sure it's a min

\[
S''(r) = 4\pi + 6\frac{4\pi}{r^3}
\]

\[
S''(2) = 4\pi + 6\frac{4\pi}{(2)^3} > 0
\]

**min. at** \(r = 2\)

\[
h = \frac{16}{r^2} = \frac{16}{4} = 4
\]

The dimensions that yield minimum S.A. are radius of 2 cm and height of 4 cm.
b) Cheapest production cost

Let \( r = \) radius \( h = \) height \( x = \) cost per cm\(^2\) of material (constant)

\[
\text{Cost} = 2x \left( \text{Area of top} \right) + x \left( \text{Area of side} \right) = 2x (2\pi r^2) + x (2\pi rh)
\]

\[
C(r) = 4\pi r^2 + 2\pi rh \quad \text{but} \quad h = \frac{16}{r^2}
\]

\[
C(r) = 4\pi r^2 + \frac{32\pi}{r}
\]

\[
C'(r) = 8\pi r - \frac{32\pi}{r^2}
\]

\[
0 = 8\pi r - \frac{32\pi}{r^2}
\]

\[
r = 2 r = \frac{8}{r^2}
\]

\[
\frac{r^2}{8} = (2r) - \frac{8}{r^2}
\]

\[
\frac{8}{r^2} = 2r
\]

\[
\frac{8}{2} = \frac{2r^3}{2}
\]

\[
r = r^3
\]

\[
r = \sqrt[3]{4}
\]

\[
h = \frac{16}{r^2} = \frac{16}{(\sqrt[3]{4})^2}
\]

A radius of \( \sqrt[3]{4}\) cm and a height of \( \frac{16}{(\sqrt[3]{4})^2}\) would minimize production costs.
A farmer wants to enclose a rectangular field with an area of 400 m². 

a) What is the smallest amount of fencing needed? 

b) What are the dimensions? 

Let \( w \) = width, \( l \) = length. 

**Perimeter** = \( 2l + 2w \) 
\[ P = 2l + 2w \]

**Area** = \( 400 \) 
\[ l \cdot w = 400 \]
\[ l = \frac{400}{w} \]

Substitute into \( P(w) = 2l + 2w \) 
\[ P(w) = 2 \left( \frac{400}{w} \right) + 2w \]
\[ P(w) = \frac{800}{w} + 2w \]

Now find \( P'(w) \): 
\[ P'(w) = -\frac{800}{w^2} + 2 \]
\[ 0 = -\frac{800}{w^2} + 2 \]
\[ \frac{800}{w^2} = 2 \]
\[ w^2 = 400 \]
\[ w = \pm \sqrt{400} \]
\[ w = \pm 20 \]
\[ w = 20 \] (Answer) 

Make sure it's a min: 
\[ f''(w) = \frac{1600}{w^3} \]
\[ f''(20) = \frac{1600}{20^3} = \text{positive} \]

Now find smallest perimeter! 
\[ l = \frac{400}{w} = 20 \]

Perimeter = \( 2l + 2w \) 
\[ = 2(20) + 2(20) \]
\[ = 40 + 40 = 80 \]

The farmer would need 80 meters of fencing with a length and width of 20 meters.