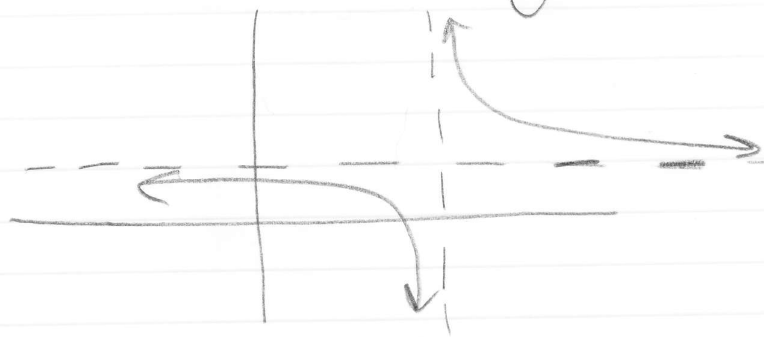
Section 3.3 Curve Sketching



- horizontal asymptotes deal with limits at ∞
- vertical asympt. deal with limits at a particular value (left hand/right hand).

11am

Vertical Asymptotes (occur when there is division by zero)

The line $x=c$ is a vertical asymptote of $f(x)$

if $\lim_{x \to c^-} f(x) = \infty$ or $(-\infty)$

or if $\lim_{x \to c^+} f(x) = \infty$ or $(-\infty)$ } Infinite limit

$\frac{\#}{\text{small}} = \infty$

ex. use calc. to find vertical asymptote(s) of $f(x) = \frac{x+2}{x-4}$ reduce? find where denom $=0$

$x-4=0$

$x=4$

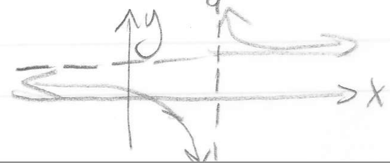
take LH & RH limit

$\to 4$

$\lim_{x \to 4^-} \frac{3.9999+2}{3.9999-4} = \frac{\#}{-(\text{small})} = -\infty$

$\lim_{x \to 4^+} \frac{4.00001+2}{4.00001-4} = \frac{\#}{+(\text{small})} = +\infty$

There is a vertical asymptote.
We take both left & right hand limit so we can find behavior of graph



Horizontal Asymptotes

The line $y=b$ is a horizontal asymptote of $y=f(x)$ if $\lim_{x\to -\infty} f(x) = b$

or

$\lim_{x\to \infty} f(x) = b$

ex) Use calculus to determine the horizontal asymptotes of $f(x) = \frac{x^2-9}{x^2+3x}$

$$\lim_{x\to\infty} \frac{x^2-9}{x^2+3x} \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{9}{x^2}}{\frac{x^2}{x^2} + \frac{3x}{x^2}} = \frac{1-0}{1+0} = 1$$

$\boxed{y=1 \text{ is a horizontal asymptote.}}$

Let's think about vertical asymptotes.

Find all $f(x) = DNE$

Denom $=0$

$x^2+3x=0$

$x(x+3)=0$

$x=0 \quad x=-3$

$$\lim_{x\to 0^-} \frac{x^2-9}{x^2+3x} = \lim_{x\to 0^-} \frac{\cancel{(x+3)}(x-3)}{x\cancel{(x+3)}} = \lim_{x\to 0^-} \frac{x-3}{x} = \frac{-0.00001-3}{-0.00001} = \frac{-\#}{-sm} = \infty$$

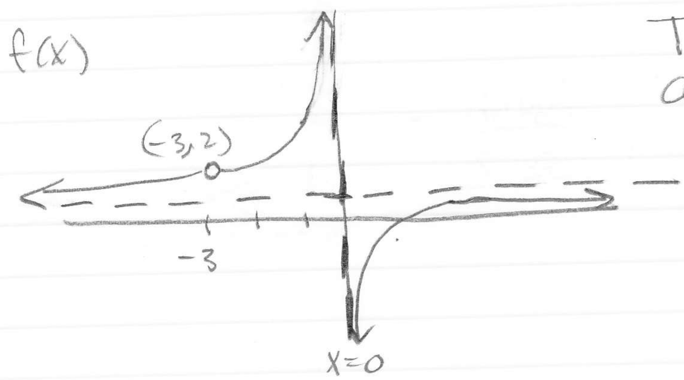
$$\lim_{x\to 0^+} \frac{x-3}{x} = \frac{0.00001-3}{0.00001} = \frac{-\#}{+small} = -\infty$$

$$\lim_{x\to -3^-} \frac{x-3}{x} = \frac{-3-3}{-3} = \frac{-6}{-3} = 2$$

$\lim_{x\to -3^+}$ same

vertical asymp. @ $x=0$

hole in graph @ $x=-3$.

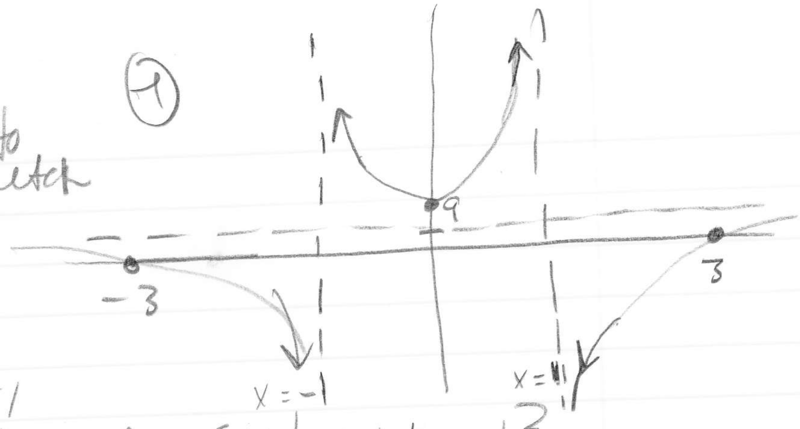


To get more accurate, find **X & y intercepts** & use & f'' test to find max/min and inflection pts.

pg 228 Procedure for sketching f using calc

1. Find domain of f(x)
2. Find and plot all intercepts
   X int ⟹ y=0 or f(x)=0 (x,0)
   y int ⟹ x=0 find f(0)=? (0,y)
3. Find all vert/horiz. asymptotes of graph & draw them on the graph
4. Find f'(x) & use it to find intervals of incr/decr. & find critical values. (Domain)
5. Find relative extrema (ordered pairs)
   Plot the max and min
6. Find f''(x) & use it to find intervals of concavity & points of inflection. Plot inflection pts with a ~~fast~~ Domain
7. Plot additional points if needed, & smooth a curve through points.

(7)



Use the 7 step process to sketch

ex. $f(x) = \dfrac{x^2-9}{x^2-1}$

1. Domain of f

$x^2-1=0$

$x^2=1 \quad x=\pm 1$

$x \neq 1 \quad x \neq -1 \qquad \text{Dom } f = \{x \mid x \neq 1, -1\}$

Plug 1's & get division by zero!

3. $\lim\limits_{x\to 1^-} \dfrac{(.9999)^2-9}{.9999^2-1} \approx \dfrac{1-9}{-\text{small}} = \dfrac{-\#}{\text{small}} = +\infty$

$\lim\limits_{x\to 1^+} \dfrac{1.00001^2-9}{(1.00001)^2-1} \approx \dfrac{-\#}{+\text{small}} = -\infty$

$\lim\limits_{x\to -1^-} \dfrac{(-1.00001)^2-9}{(-1.00001)^2-1} \approx \dfrac{-\#}{+\text{small}} = -\infty$

$\lim\limits_{x\to -1^+} \dfrac{(-.9999)^2-9}{(-.9999)^2-1} \approx \dfrac{-\#}{-\text{small}} = +\infty$

Vertical asymptotes @ $x=1$ & $x=-1$

$\lim\limits_{x\to\pm\infty} \dfrac{x^2-9}{x^2-1} = \lim\limits_{x\to\infty} \dfrac{\frac{x^2}{x^2}-\cancel{\frac{9}{x^2}}}{\frac{x^2}{x^2}-\cancel{\frac{1}{x^2}}} = \dfrac{1}{1}$

Horiz. Asymp. @ $y=1$

2. Intercepts

y int $\Rightarrow x=0$

$f(0) = \dfrac{0-9}{0-1} = \dfrac{-9}{-1} = 9 \qquad (0,9)$

x int $\Rightarrow f(x)=0 \qquad (3,0)\ (-3,0)$

$0 = \dfrac{x^2-9}{x^2-1}$

$0 = x^2-9$

$9 = x^2$

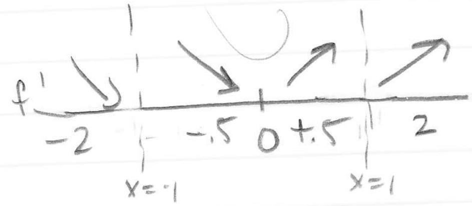
$x = \pm\sqrt{9}$

$\boxed{x = \pm 3}$

4. $f'(x) = \dfrac{(x^2-1)(2x)-(x^2-9)(2x)}{(x^2-1)^2}$

$f'(x) = \dfrac{\cancel{2x^3}-2x-\cancel{2x^3}+18x}{(x^2-1)^2}$

$f'(x) = \dfrac{16x}{(x^2-1)^2} \qquad 0 = 16x \qquad \boxed{x=0}$ C.V.



Increasing
$(0,1) \cup (1,\infty)$
Decreasing
$(-\infty,-1) \cup (-1,0)$

$f'(x) = \dfrac{16x}{(x^2-1)^2}$

$f'(-2) = -$
$f'(-.5) = -$
$f'(.5) = +$
$f'(2) = +$

5. min @ $x=0$
$f(0) = 9$
$(0,9)$

Plot it! already

6. $f''(x) = \dfrac{(x^2-1)^2\,16 - 16x\,(2\,(x^2-1)'\,(2x))}{((x^2-1)^2)^2}$

$= \dfrac{16\cancel{(x^2-1)}\left[(x^2-1) - x\cdot 4x\right]}{(x^2-1)^{\cancel{4}\,3}}$

$= \dfrac{16\left[x^2-1-4x^2\right]}{(x^2-1)^3}$

$\boxed{f''(x) = \dfrac{16(-3x^2-1)}{(x^2-1)^3}}$

$0 = 16(-3x^2-1)$

$\cancel{0=16}$ $\quad -3x^2-1=0$

$\dfrac{-3x^2}{-3} = \dfrac{1}{-3}$

$x^2 = -\dfrac{1}{3}$

$\cancel{x = \pm\sqrt{-\tfrac{1}{3}}}$

f'' (DNE)
$(x^2-1)^3 = 0$
$x^2-1=0$
$x = \pm 1$

There are no inflection pts

$f'(0) = \dfrac{16(-3(0)-1)}{(0-1)^3}$

$= \dfrac{16(-1)}{-1}$

$= 16$

$f'(\mp 2) = \dfrac{-3(-2)^2-1}{((-2)^2-1)^3} = \dfrac{-3(4)-1}{(4-1)^3} = \dfrac{-}{+}$

Note
$\sqrt[3]{x}$ Domain
$\mathbb{R}$
index is odd

ex. Use the 7 step process to sketch
$f(x) = x^{2/3}$ $\sqrt[3]{x^2}$

① Domain of f
$\mathbb{R}$

② Intercepts
$x^{2/3} = 0$
$\sqrt[3]{x^2} = 0$
$x^2 = 0$
$x = 0$
x/y int $\boxed{(0,0)}$

③ Asymptotes
No places where f is undefined (No division by zero)
so no vertical asymptotes

$\lim_{x \to \infty} x^{2/3} = \infty$ No horiz. asymptotes

④ Find $f'(x)$ & CV's (Incr/Decr.)
$f'(x) = \frac{2}{3}x^{2/3 - 3/3} = \frac{2}{3}x^{-1/3}$

$f'(x) = \frac{2}{3\sqrt[3]{x}}$

$0 = \frac{2}{3\sqrt[3]{x}}$ $0 \neq 2$

Now Check: $f'(x) = DNE$ & $f(x)$ exists.
$f'(x) = DNE$ when $x = 0$
Does $f(0)$ exist? yes
$f(0) = 0$ so $\boxed{(0,0) \text{ is a critical point}}$

⑤ f' − − − − + + + + +
−1 0 1

$\boxed{\text{minimum @ } (0,0)}$

$\boxed{\begin{array}{l}\text{Incr. } (0,\infty) \\ \text{Decr. } (-\infty,0)\end{array}}$

$f'(-1) = 2/(3\sqrt[3]{-1}) = -$
$f'(1) = 2/(3\sqrt[3]{1}) = +$

⑥ Find $f''(x)$ & concavity & inflection points

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$f''(x) = \frac{2}{3}\left(\frac{-1}{3}\right)x^{-\frac{1}{3}-\frac{3}{3}}$$

$$= -\frac{2}{9}x^{-4/3}$$

$$f''(x) = \frac{-2}{9\sqrt[3]{x^4}}$$

$$0 = \frac{-2}{9\sqrt[3]{x^4}}$$

$0 \neq -2$ No pts of inflection

$f''(x) = DNE$ when $x=0$

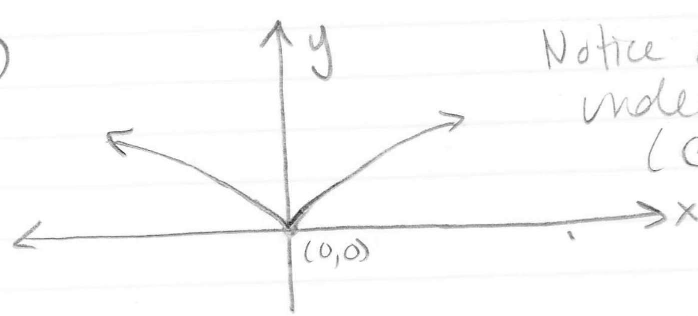
f'' − − − − − − − −

−1 0 1

$$f''(\pm 1)\ \frac{-2}{9\sqrt[3]{(-1)^4}} = \frac{-}{+} = -$$

$\boxed{\text{Concave down } (-\infty,0)\cup(0,\infty)}$

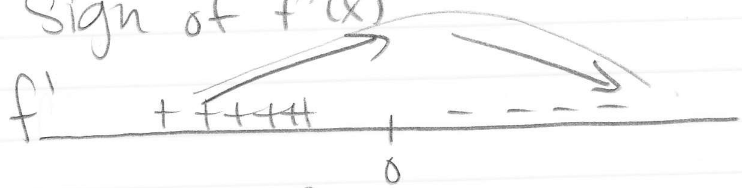
⑦



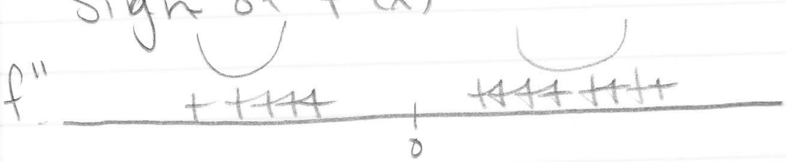
Notice derivative is undefined @ $x=0$ (cusp)

Sketching Graphs

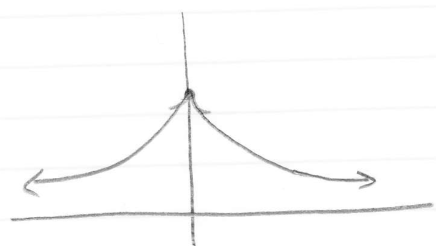
ex) Sign of $f'(x)$



Sign of $f''(x)$



Sketch of f



ex) Sign of $f'(x)$

Sign of $f''(x)$

OR