Section 2.6 - Implicit Differentiation

Suppose a function defines y implicitly as a function of x, you may try to solve for y first, then differentiate. This option is not always practical.

Implicitly defined equation: $y+4x=6+5y$

Explicit (solved for y): Easy example

$$y+4x=6+5y$$
$$4x=6+4y$$
$$4x-6=4y$$
$$x-\frac{3}{2}=y$$

$$y=x-\frac{3}{2}$$
$$y'=1$$

What about the implicit equation:

$x^2y+y^3=2x$ we have to be able to differentiate implicitly...

11am

Implicit Differentiation

① Differentiate both sides of a given equation w.r.t. x (must use chain rule on all terms with a y in them since y is itself a function of x)

② Solve for $\frac{dy}{dx}$ (or $f'(x)$ or y')

③ There may be x's y's in $\frac{dy}{dx}$.

example:

$$5x+3y=2$$
$$3y=-5x+2$$
$$y=-\frac{5}{3}x+\frac{2}{3}$$
$$\boxed{y'=-5/3}$$

$$\frac{d}{dx}5x+\frac{d}{dx}3y=\frac{d}{dx}2$$
$$5+3\frac{dy}{dx}=0$$
$$\boxed{\frac{dy}{dx}=-\frac{5}{3}}$$

ex. Solve for y 1st

$2x^2+y^3=5$

$y^3=5-2x^2$

$y=\sqrt[3]{5-2x^2}$

$y=(5-2x^2)^{1/3}$

$y'=\frac{1}{3}(\quad)^{1/3-3/3}(\quad)'$

$=\frac{1}{3}(5-2x^2)^{-2/3}(-4x)$

$\boxed{y'=\frac{-4x}{3(\sqrt[3]{5-2x^2})^2}}$

Implicitly

$2x^2+y^3=5$

$\frac{d}{dx}2x^2+\frac{d}{dx}y^3=\frac{d}{dx}5$

$4x+3y^2\frac{dy}{dx}=0$

$3y^2\frac{dy}{dx}=-4x$

$\boxed{\frac{dy}{dx}=\frac{-4x}{3y^2}}$

How do we know if they are the same?

Plug y= ~ into implicit derivative!

$\frac{dy}{dx}=\frac{-4x}{3y^2}$

$=\frac{-4x}{3(\sqrt[3]{5-2x^2})^2}$

Same!! :)

Examples of derivatives of terms with y in them

ex. $(4y)'=4\frac{dy}{dx}$ or $4y'$

ex. $(6y^3)'=18y^2\frac{dy}{dx}$ or $18y^2y'$

ex. $(y^{-2})'=-2y^{-3}\frac{dy}{dx}$ or $-2y^{-3}y'$

ex. $(\sqrt{y})'=(y^{1/2})'=\frac{1}{2}y^{-1/2}\frac{dy}{dx}$ or $\frac{1}{2}y^{-1/2}y'$

ex. $(x\cdot y)'=x\frac{dy}{dx}+y(1)$ OR $=xy'+y$

must use product rule!

ex. $((2+y)^3)'=3(2+y)^2\cdot(1\frac{dy}{dx}$ OR $=3(2+y)^2\cdot y'$

must use chain rule

ex. $(2+y)^3=5x^2$ diff. Impl

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ex. Find derivative in terms of x and y and then use it to find the slope of the tangent line to the curve $x^2+y^2=25$ at the point $(3,-4)$

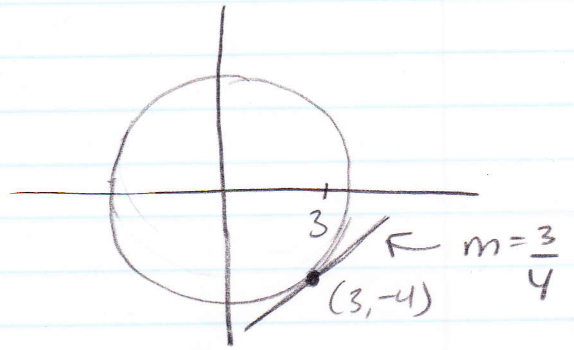
$$\frac{d}{dx}x^2+\frac{d}{dx}y^2=\frac{d}{dx}25$$

$$2x+2y\cdot y'=0$$

$$2yy'=-2x$$

$$y'=\frac{-2x}{2y}$$

$$y'=\frac{-x}{y}$$



$$y'\Big|_{x=3}=\frac{-(3)}{-4}=\boxed{\frac{3}{4}}$$

What if they didn't give us y? Use the original equation to find y.

→ think about dom of f(x)

What about at $(-5,0)$? $y'=\frac{5}{0}$ = UND vertical tangent

ex. Find y' using implicit differentiation

Intro y'

$$x^3+y^3=x\cdot y \leftarrow \text{prod. rule } (x\cdot f(x))$$

two diff. functions of x!

$$3x+3y^2y'=xy'+y\cdot 1$$

$$3y^2y'-xy'=y-3x$$

$$y'(3y^2-x)=y-3x$$

$$\boxed{y'=\frac{y-3x}{3y^2-x}}$$

ex. Find y'

$$x+\frac{1}{y}=5$$

$$x+y^{-1}=5$$

$$1+-1y^{-2}\cdot y'=0$$

$$-y^{-2}\cdot y'=-1$$

$$y'=\frac{-1}{-y^{-2}}=\frac{1}{y^{-2}}=y^2$$

Do this one!

ex. $(x-2y)^2 = y$ Could use chain rule, but it's ugly! FOIL 1st!

$$x^2 - 4xy + 4y^2 = y$$

$$2x - 4[x \cdot 1y' + y(1)] + 8yy' = 1y'$$

$$2x - 4[xy' + y] + 8yy' = y'$$

$$2x - 4xy' - 4y + 8yy' = y'$$

$$-4xy' + 8yy' - y' = 4y - 2x$$

$$y'(-4x + 8y - 1) = 4y - 2x$$

$$\boxed{y' = \frac{4y - 2x}{(-4x + 8y - 1)}}$$

ex. Find the equation of the tangent line at $x = \frac{1}{2}$

if lost on time $(\frac{1}{2}, 1)$

$$\frac{1}{x} + \sqrt{y} = 1$$

$$x^{-1} + y^{1/2} = 1$$

$$-1x^{-2} + \frac{1}{2}y^{-1/2}\, y' = 0$$

$$-\frac{1}{x^2} + \frac{1}{2\sqrt{y}}\, y' = 0$$

$$2\sqrt{y}\left(\frac{1}{2\sqrt{y}}\, y'\right) = \left(\frac{1}{x^2}\right)2\sqrt{y}$$

$$\boxed{y' = \frac{2\sqrt{y}}{x^2}}$$

If $x = \frac{1}{2}$ $y = ?$

$$\frac{1}{(\frac{1}{2})} + \sqrt{y} = 1$$

$$2 + \sqrt{y} = 1$$

$$\sqrt{y} = 1 - 2$$

$$\sqrt{y} = -1$$

$$y = 1$$

$(\frac{1}{2}, 1)$

$$y'\Big|_{\substack{x=\frac{1}{2} \\ y=1}} = \frac{2\sqrt{1}}{(\frac{1}{2})^2} = \frac{2}{\frac{1}{4}} = 2 \cdot 4 = 8$$

$m = 8$ $(\frac{1}{2}, 1)$

$$y - 1 = 8(x - \tfrac{1}{2})$$

$$y - 1 = 8x - 4$$

$$y = 8x - 4 + 1$$

$$\boxed{y = 8x - 3}$$