Section 2.6 - Implicit Differentiation

Suppose a function defines \( y \) implicitly as a function of \( x \), you may try to solve for \( y \) first, then differentiate. This option is not always practical.

Implicitly defined equation: \( y + 4x = 6 + 5y \)

Explicit (solved for \( y \)): Easy example

\[
\begin{align*}
y + 4x &= 6 + 5y \\
x &= 6 + 4y \\
x &= 6 + 4y \\
x &= \frac{3}{2} = y
\end{align*}
\]

\[
y = x - \frac{3}{2}
\]

\[
y' = \frac{1}{2}
\]

What about the implicit equation:

\( x^2y + y^3 = 2x \) we have to be able to differentiate implicitly...

Implicit Differentiation

1. Differentiate both sides of a given equation w.r.t. \( x \) (must use chain rule on all terms with a \( y \) in them since \( y \) is itself a function of \( x \)).

2. Solve for \( \frac{dy}{dx} \) (or \( f'(x) \) or \( y' \))

3. There may be \( x \)'s \( y \)'s in \( \frac{dy}{dx} \)

Example:

\[
\begin{align*}
5x + 3y &= 2 \\
3y &= -5x + 2 \\
y &= -\frac{5}{3}x + \frac{2}{3}
\end{align*}
\]

\[
y' = -\frac{5}{3}
\]
ex. Solve for \( y \) 1st

\[
2x^2 + y^3 = 5
\]

\[
y^3 = 5 - 2x^2
\]

\[
y = \sqrt[3]{5 - 2x^2}
\]

\[
y' = \frac{1}{3} (5 - 2x^2)^{-2/3} (-4x)
\]

\[
y' = \frac{-4x}{3(3\sqrt{5 - 2x^2})^2}
\]

How do we know if they are the same?

Plug \( y = \) into implicit derivative!

\[
\frac{dy}{dx} = \frac{-4x}{3y^2}
\]

Same!!

Examples of derivatives of terms with \( y \) in them

ex. \((4y)' = 4 \frac{dy}{dx} \) or \( 4y' \)

ex. \((6y^3)' = 18y^2 \frac{dy}{dx} \) or \( 18y^2 y' \)

ex. \((y^{-2})' = -2y^{-3} \frac{dy}{dx} \) or \(-2y^{-3} y' \)

ex. \((y^{1/2})' = \frac{1}{2} y^{-1/2} \frac{dy}{dx} \) or \( \frac{1}{2} y^{-1/2} y' \)

ex. \((2y^3)' = 3(2y^2) \frac{dy}{dx} \) or \( 3(2y^2) y' \)

ex. \((2y+y^3)' = x \frac{dy}{dx} + y (1) \)

ex. \((4y^3)' = 12y^2 \frac{dy}{dx} \) or \( 12y^2 y' \)

ex. \((2+y)^3)' = 3(2+y)^2 \frac{dy}{dx} \) or \( 3(2+y)^2 y' \)

ex. \((2+y^3)' = 5x^2 \text{ diff. imp.} \)
2.6 (3)

Ex. Find derivative in form of \( x \) and \( y \) and then use it to find the slope of the tangent line to the curve \( \frac{x^2 + y^2}{dx} = 25 \) at the point \((3, -4)\).

\[
\frac{d}{dx} x^2 + \frac{dy}{dx} y^2 = \frac{d}{dx} 25 \\
2x + 2yy' = 0 \\
y' = \frac{-2x}{2y} \\
y' = \frac{-x}{y}
\]

\[
y' \bigg|_{x=3} = \frac{-13}{4} = \frac{3}{4}
\]

What if they didn't give us \( y \)? Use the original equation to find \( y' \):

ex. Find \( y' \) using implicit differentiation:

\( x^3 + y^3 = x \cdot y \)

\( 3x + 3y^2 y' = xy' + y' \cdot 1 \)

\( 3y^2 y' - xy' = y - 3x \)

\( y'(3y^2 - x) = y - 3x \)

\( y' = \frac{y - 3x}{3y^2 - x} \)

ex. Find \( y' \):

\( x + \frac{1}{y} = 5 \)

\( x + y^{-1} = 5 \)

\( 1 - y^{-2} \cdot y' = 0 \)

\( -y^{-2} \cdot y' = -1 \)

\( y' = \frac{1}{y^2} = \frac{1}{y^{-2}} = y^2 \)
ex. \((x-2y)^2 = y\)

FOIL 1st!

\[2x - 4[x \cdot y' + y(1)] + 8yy' = y'\]

\[2x - 4[x y' + y] + 8yy' = y'\]

\[2x - 4xy' - 4y + 8yy' = y'\]

\[-4xy' + 8yy' - y' = 4y - 2x\]

\[y'(-4x + 8y - 1) = 4y - 2x\]

\[y' = \frac{4y - 2x}{-4x + 8y - 1}\]

ex. Find the equation of the tangent line at \(x = \frac{1}{2}\)

\[\frac{1}{x} + \sqrt{y} = 1\]

\[x^{-1} + y_{1/2} = 1\]

\[-1x^{-2} + \frac{1}{2} y^{-1/2} y' = 0\]

\[-\frac{1}{x^2} + \frac{1}{2 \sqrt{y}} y' = 0\]

\[2\sqrt{y} \left( \frac{1}{2\sqrt{y}} \right) y' = (\frac{1}{x^2}) 2\sqrt{y}\]

\[y' = \frac{2\sqrt{y'}}{x^2}\]

\[y' \bigg|_{x=\frac{1}{2}} = \frac{2\sqrt{\frac{1}{4}}}{\left(\frac{1}{2}\right)^2}\]

\[y = \frac{2}{\frac{1}{2}} = 2 \cdot 4 = 8\]

\[m = 8 \ (\frac{1}{2}, 1)\]

\[y - 1 = 8 \ (x - \frac{1}{2})\]

\[y - 1 = 8x - 4\]

\[\frac{y}{8x - 4 + 1}\]

\[y = 8x - 3\]