Section 2.4 - The Chain Rule

<u>The chain rule</u>

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a diff. function of x, then the composite function $y = f(g(x))$ is differentiable and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or

$$\frac{dy}{dx} = f'(g(x))\,g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \frac{\cancel{du}}{dx} = \frac{dy}{dx}$$

ex. Find $\frac{dy}{dx}$ if $y = (x^2+2)^3$ ← inner

outer → $y(u) = u^3$ and $u(x) = x^2+2$

$dy/du = 3u^2$ $\quad du/dx = 2x$

By the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$= 3u^2 \cdot 2x$ Then plug in $u = x^2+2$

$$= 3(x^2+2)^2 \cdot 2x$$
$$= 3(x^4+4x^2+4) \cdot 2x$$
$$= (3x^4+12x^2+12) \cdot 2x$$
$$= \boxed{6x^5 + 24x^3 + 24x}$$

ex. Use chain rule to compute $\frac{dy}{dx}$

$y = 1 - 3u^2$, $u = 3 - 2x$

then $\frac{dy}{du} = -6u$ $\quad \frac{du}{dx} = -2$

plug in u!

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -6u \cdot -2 = 12u = 12(3-2x)$$
$$= \boxed{36 - 24x}$$

11am

ex. Use chain rule to compute $\frac{dy}{dx}$ for the given value of x.

$$y = (x^2-1)^3 - 3(x^2-1)^2 + 6(x^2-1) - 5 \qquad x=1$$

outter

$$y = u^3 - 3u^2 + 6u - 5$$

$$\frac{dy}{du} = 3u^2 - 6u + 6$$

Inner

$$u = x^2-1$$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (3u^2 - 6u + 6)(2x)$$

Method 1: substitute $u = x^2-1$ back in (as we have been)

$$= (3(x^2-1)^2 - 6(x^2-1) + 6)(2x)$$

$$= [3(x^4 - 2x^2 + 1) - 6x^2 + 6 + 6]\,2x$$

$$= [3x^4 - 6x^2 + 3 - 6x^2 + 12]\,2x$$

$$= [3x^4 - 12x^2 + 15]\,2x$$

$$= 6x^5 - 24x^3 + 30x$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 6(1)^5 - 24(1)^3 + 30(1)$$

$$= 6 - 24 + 30$$

$$= 36 - 24$$

$$= \boxed{12}$$

1pm

OR

Method 2: find u when x=1 & subst.

$$u = x^2-1$$

$$x=1 \Rightarrow u = 1^2 - 1$$

$$u = 0$$

Same! Aha!

$$\frac{dy}{dx} = (3u^2 - 6u + 6)\,2x$$

$$\left.\frac{dy}{dx}\right|_{\substack{u=0\\x=1}} = (3(0)^2 - 6(0) + 6)\,2(1) = 6 \cdot 2 = \boxed{12}$$

Method 2 is shorter, but sometimes we want an actual equation for dy/dx in terms of x only, then you must use method ~~1~~.

ex. Find $\left.\frac{dy}{dx}\right|_{x=3}$ for $y = \sqrt{u}$ $= u^{1/2}$ $\quad u = x^2 - 2x + 6$

$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ $\quad \frac{du}{dx} = 2x - 2$

$x = 3$

$u = 3^2 - 2(3) + 6$
$= 9 - \cancel{6} + \cancel{6}$
$u = 9$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{2\sqrt{u}} \cdot (2x-2)$$

$$\left.\frac{dy}{dx}\right|_{\substack{u=9\\x=3}} = \frac{1}{2\sqrt{9}}(2(3)-2)$$
$$= \frac{1}{6}(6-2) = \frac{4}{6} = \boxed{\frac{2}{3}}$$

Slope of tan. line, still, of the composite function y.

Application?

$y = f(g(x))$ ← outter function ; inner function

$\frac{dy}{dx} = f'(g(x))\, g'(x)$

"The deriv. of the outter function evaluated at the inner function times the deriv. of the inner function"

ex. $f(x) = (x^2+3x+2)^6$

$f(x) = (\square)$ ← outter function

$f'(x) = 6(\square)^5 \cdot (\square)'$ $\quad \square = x^2+3x+2$, $\square' = 2x+3$

$= 6(x^2+3x+2)^5 \; (2x+3)$

$= \boxed{(x^2+3x+2)^5(12x+18)}$

2.4
The Generalized Power Rule
For any real # n & differentiable function h,

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1} \cdot \frac{d}{dx}[h(x)]$$

or $h'(x) = n[h(x)]^{n-1} \cdot (h'(x))$

or $f(x) = (\text{Stuff})^6$
$f'(x) = 6(\text{Stuff})^5 \cdot (\text{Stuff})'$

Show an example w/ gen'l power & chain rule

ex. (2) Differentiate
$y = (x^5 - 4x^3 - 7)^8$

$$\boxed{\frac{dy}{dx} = 8(x^5 - 4x^3 - 7)^7 \cdot (5x^4 - 12x^2)}$$

ex. Diff.
(3) $f(x) = \frac{1}{(2x+3)^5} = (2x+3)^{-5}$

$f'(x) = -5(2x+3)^{-6}(2x+3)'$
$= -5(2x+3)^{-6}(2)$
$= \frac{-10}{(2x+3)^6}$

(1) ex. $f(x) = \sqrt{3x+2}$
$= (3x+2)^{1/2}$
$f'(x) = \frac{1}{2}(\quad)^{-1/2}(\quad)'$
$= \frac{1}{2}(3x+2)^{-1/2}(3)$
$\boxed{= \frac{3}{2\sqrt{3x+2}}}$

Combine rules
ex. $f(x) = \underset{f}{(3x+1)^4}\underset{g}{(5x-3)^2}$ "Big Picture" → Product

$f = (3x+1)^4$
$f' = 4(3x+1)^3(3) = 12$
$g = (5x-3)^2$
$g' = 2(5x-3)^1(5)$
$= 10(5x-3)$

$f'(x) = fg' + gf'$
$= (3x+1)^4[10(5x-3)] + (5x-3)^2[12(3x+1)^3]$
$= (3x+1)^3(5x-3)[(3x+1)\cdot 10 + (5x-3)\cdot 12]$
$= (3x+1)^3(5x-3)[30x+10+60x-36]$
$= (3x+1)^3(5x-3)[90x-26]$
$\boxed{f'(x) = (3x+1)^3(5x-3)(90x-26)}$