Section 2.4 - The Chain Rule

The chain rule

If \( y = f(u) \) is a differentiable function of \( u \) and \( u = g(x) \) is a differentiable function of \( x \), then the composite function \( y = f(g(x)) \)

is differentiable and

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

or

\[
\frac{dy}{dx} = f'(g(x)) \cdot g'(x)
\]

Ex. Find \( \frac{dy}{dx} \) if \( y = (x^2+2)^3 \).

Let \( y(u) = u^3 \) and \( u(x) = x^2+2 \)

By the chain rule,

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

= \( 3u^2 \cdot 2x \)  \( \text{then plug in } u = x^2+2 \)

\[
= 3(x^2+2)^2 \cdot 2x
\]

\[
= 3(x^4+4x^2+4) \cdot 2x
\]

\[
= \frac{3x^4+12x^2+12 \cdot 2x}{6x^3+24x^3+24x}
\]

Ex. Use chain rule to compute \( \frac{dy}{dx} \)

\( y = 1-3u^2, \ u = 3-2x \)

Then \( \frac{dy}{du} = -6u, \ \frac{du}{dx} = -2 \)

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -6u \cdot -2 = 12u = 12(3-2x)
\]

\[
= 12(3-2x)
\]
Use chain rule to compute \( \frac{dy}{dx} \) for the given value of \( x \).

**Outer:**
\[ y = (x^2-1)^3 - 3(x^2-1)^2 + 6(x^2-1) - 5 \]

**Inner:**
\[ u = x^2 - 1 \]
\[ \frac{du}{dx} = 2x \]
\[ \frac{dy}{du} = 3u^2 - 6u + 6 \]

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]
\[ = (3u^2 - 6u + 6)(2x) \]

**Method 1:** Substitute \( u = x^2 - 1 \) back in (as we have been)

\[ = 3(x^2 - 1)^2 - 6(x^2 - 1) + 6 \cdot 2x \]
\[ = 3x^4 - 12x^2 + 12 - 6x^2 + 6 + 12 \]
\[ = 3x^4 - 18x^2 + 18 \]
\[ = 6x^2 - 24x^2 + 30x \]

\[ \left. \frac{dy}{dx} \right|_{x=1} = 6(1)^5 - 24(1)^3 + 30(1) \]
\[ = 6 - 24 + 30 \]
\[ = 2 \]

OR

**Method 2:** Find \( u \) when \( x = 1 \) & subst.
\[ u = x^2 - 1 \]
\[ x = 1 \Rightarrow u = 1^2 - 1 \]
\[ u = 0 \]

\[ \frac{dy}{dx} = (3u^2 - 6u + 6)2x \]
\[ \frac{dy}{dx} \bigg|_{u=0} = 6(6^2 - 6(0) + 6)2(1) = 6 \cdot 2 \cdot 12 \]
Method 2 is shorter, but sometimes we want an actual equation for $\frac{dy}{dx}$ in terms of $x$ only, then you must use method 1.

**Ex.** Find $\frac{dy}{dx}$ for $y = \sqrt{u}$, $u = x^2 - 2x + 6$.

\[
\frac{dy}{du} = \frac{1}{2\sqrt{u}} \
\frac{du}{dx} = 2x - 2
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} (2x-2)
\]

At $x = 3$:

\[
\frac{dy}{dx} \bigg|_{u=9} = \frac{1}{2\sqrt{9}} (2(3)-2) = \frac{1}{3} (6-2) = \frac{4}{3} = \frac{2}{3}
\]

Application?

The derivative of the outer function evaluated at the inner function times the derivative of the inner function.

**Ex.** $f(x) = (x^2 + 3x + 2)^6$.

\[
f'(x) = 6(x^2 + 3x + 2)^5 (2x + 3)
\]

\[
= 6 \left( \frac{x^2 + 3x + 2}{(2x + 3)} \right)^5 (2x + 3)
\]

\[
= \left( \frac{x^2 + 3x + 2}{(2x + 3)} \right)^5 (12x + 18)
\]

Slope of tan. line, still, of the composite function $y$. 

> outer function

> inner function

> The deriv. of the outer function evaluated at the inner function times the deriv. of the inner function.
The Generalized Power Rule
For any real \( n \) and differentiable function \( h \),
\[
\frac{d}{dx} [h(x)]^n = n [h(x)]^{n-1} \cdot \frac{d}{dx} [h(x)]
\]
or\[h'(x) = n [h(x)]^{n-1} (h'(x))]

or\[f'(x) = (\text{Stuff})^5 \\
= 6 (\text{Stuff})^5 \cdot (\text{Stuff})'
\]

Example Differentiate
\[y = (x^5 - 4x^3 - 7)^8\]
\[
\frac{dy}{dx} = 8 (x^5 - 4x^3 - 7)^7 \cdot (5x^4 - 12x^2)
\]

Example, Diff. 3 \[f(x) = \frac{1}{(2x+3)^5} = (2x+3)^{-5}\]
\[
f'(x) = -5 (2x+3)^{-6} (2x+3)'
\]
\[
= -5 (2x+3)^{-6} (2)
\]
\[
= \frac{-10}{(2x+3)^6}
\]

Combine rules
Example, \[f(x) = (3x+1)^4 (5x-3)^2 \]

Apply Product Rule
\[
f = (3x+1)^4 \\
f' = 4(3x+1)^3 (3) = 12 \\
g = (5x-3)^2 \\
g' = 2(5x-3) \cdot (5) = 10 (5x-3)
\]
\[
f'(x) = fg' + gf' = (3x+1)^4 \left[ 10(5x-3) + (5x-3)^2 \right] / 12 (3x+1)^3
\]
\[
f'(x) = (3x+1)^3 (5x-3) (90x - 26)
\]