Section 2.3 Product and Quotient Rules
and Higher Order Derivatives

We have sum/difference rules...but what about
If \( f(x) = x^3 \) and \( g(x) = x \) then
\( f'(x) = 3x^2 \) and \( g'(x) = 4x^3 \), so
so \( f'(x)g'(x) = (3x^2)(4x^3) \)

\[ = 12x^5 \]

But...
\( f(x)g(x) = (x^3)(x^4) \)
\( = x^7 \)
and \( [f(x)g(x)]' = 7x^6 \)

We knew this is okay, but there's
a special product rule.

The Product Rule
\[ \frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x) \]

or
\( (f \cdot g)' = fg' + gf' \) "1st times deriv of 2nd + 2nd times deriv of 1st" proof pg. 138

Let's try again...'
f(x) = x^3 \quad g(x) = x^4

[\[f(x)g(x)]' = x^3(4x^3) + x^4(3x^2) \]
\[ = 4x^6 + 3x^6 \]
\[ = 7x^6 \]

Typically we have to use the product rule if
and can't check as we did above.
\[ f(x) = 2x(x-1) \quad \text{easy just mult act \& use power rule.} \]

\[ f(x) = 2x^2 - 2x \]

\[ f'(x) = 4x - 2 \]

What about uglier ones?

\[ h(x) = (3x^4 + 6x - 2)(5-x) \quad \text{mult act? easier to use} \]

\[ f(x) \quad g(x) \quad \text{product rule.} \]

Rule: \[ h'(x) = f \cdot g' + g \cdot f' \]

\[
\begin{align*}
   f &= 3x^4 + 6x - 2 \\
   g &= 5-x \\
   f' &= 12x^3 + 6 \\
   g' &= -1
\end{align*}
\]

\[ h'(x) = -15x^4 + 60x^3 - 12x + 32 \]

Suit:

\[ y = (5x^3 + 4\sqrt{x} + x)(\frac{1}{x} - \frac{x}{3}) \]

Take derivative \& do not simplify!

\[ y' = (5x^3 + 4\sqrt{x} + x)(-\frac{1}{x^2} - \frac{1}{3}) + \left(\frac{1}{x} - \frac{x}{3}\right)(15x^2 + \frac{4}{2\sqrt{x}} + 1) \]

Suit ex. \[ f(x) = \frac{1}{2}(x^4 - 3x^2 + 1) \]

\[ f' = x^4 - 3x^2 + 1 \]

\[ f' = yx^3 - 6x \]

\[ g' = \frac{1}{x} - 2x^{-2} \]

\[ g' = \frac{1}{3} - 2x^{-2} \]

\[ g' = \frac{1}{3} - \frac{2}{x^2} \]

\[ g' = \frac{1}{3} - \frac{2}{x^2} \]

\[ g' = \frac{1}{3} - \frac{2}{x^2} \]

\[ g' = \frac{1}{3} - \frac{2}{x^2} \]

\[ g' = \frac{1}{3} - \frac{2}{x^2} \]

\[ g' = \frac{1}{3} - \frac{2}{x^2} \]
For the curve 
\[ y = (x-1)(x^2-8x+7) - x + 5 \]

a) Find \( \frac{dy}{dx} \)

b) Find the equation of the tangent line at \( x = 2 \).

c) Find all points where the tangent line is horizontal.

\[ a) \frac{dy}{dx} = (x-1)(2x-8) + (x^2-8x+7)(1) \]
\[ = 2x^2 - 8x - 2x + 8 + x^2 - 8x + 7 \]
\[ = 3x^2 - 18x + 15 \]
\[ \frac{dy}{dx} = 3x^2 - 18x + 15 \]

b) \( x = 2 \) \[ \Rightarrow f(2) = (2-1)(2^2-8(2)+7) = 4 - 16 + 7 = 11 - 16 = -5 \]
\[ m = \left. \frac{dy}{dx} \right|_{x=2} = 3(2)^2 - 18(2) + 15 \]
\[ = 3(4) - 36 + 15 \]
\[ = 12 - 36 + 15 \]
\[ = -27 + 15 \]
\[ = -12 \]
\[ y + 5 = -12(x - 2) \]
\[ y + 5 = -12x + 24 \]
\[ y = -12x + 19 \]
\[ y = 0 \]
\[ y = -9x + 13 \]

\[ x = \frac{5}{3} \]
\[ f(5/3) = (5/3)^2 - 8(5/3) + 7 = 4 \]
\[ x = 1 \]
\[ f(1) = (1)(1) - 8 + 7 = 0 \]

Horizontal tangent lines occur at \((5/3, 2)\) and \((1, 0)\).
Quotient Rule

Common error \((f/g)' = f'/g'\) Not!

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}
\]

Or

\[
\left( \frac{f}{g} \right)' = \frac{gf' - fg'}{g^2}
\]

Numeration trick

\[
\left( \frac{h}{10} \right)' = \frac{10h' - h(10)}{100}
\]

Ex. \(Q(x) = \frac{2}{x}\)

\[
Q'(x) = \frac{x \cdot 0 - 2(1)}{x^2} = \frac{-2}{x^2}
\]

Ex. \(f(x) = (2x-1)(x+3)\)

\[
x = (x+1)^2
\]

Ex. \(g(t) = \frac{t^2 + 1E}{2t+5}\)

\[
g'(t) = \frac{(2+1E)(2t + \frac{1}{2E}) - (t^2 + 1E)(2)}{(2t+5)^2}
\]
ex. cont

\[ g'(t) = \frac{4t^2 + 2t + \frac{5}{2t} + \frac{5}{2t} - 2t^2 + 2t}{(2t+5)^2} \]

\[ = \frac{2t^2 + 10t + \frac{5}{2t} + \frac{5}{2t} + 2t}{(2t+5)^2} \]

\[ = \frac{2t^2 + 10t + \frac{5}{2t} + \frac{5}{2t} + 2t}{(2t+5)^2} \]

\[ \left(\frac{2t}{2t+5}\right) \]

\[ \text{clean up! Algebra} \]

8. Find the equation of the tangent line to the given curve at \( x = x_0 \)

\[ y = \frac{x+7}{5-2x} \]

\( x_0 = 0 \)

\[ y' = \frac{(5-2x)(1)-(x+7)(-2)}{(5-2x)^2} \]

\[ = \frac{5-2x+2x+14}{(5-2x)^2} \]

\[ y'' = \frac{19}{(5-2x)^2} \]

\( x = 0 \)

\[ y' \big|_{x=0} = \frac{19}{25} = m \]

\[ y \big|_{x=0} = \frac{7}{5} \]

\[ y - \frac{7}{5} = \frac{19}{25} (x-0) \]

\[ y = \frac{19}{25} x + \frac{7}{5} \]
Applications
Concentration of a pain-killer in the bloodstream t hours after ingesting is given by

\[ C(t) = \frac{2t}{3t^2 + 16} \]

+ in hrs

a) At what rate is the concentration changing after 1 hr?
b) What happens to the concentration in the long run \((t \to \infty)\)?
c) When does the concentration start to decrease?

\( C'(t) = \frac{(3t^2 + 16)(2) - (2t)(6t)}{(3t^2 + 16)^2} \)

\[ C'(t) = \frac{6t^2 + 32 - 12t^2}{(3t^2 + 16)^2} \]

\[ = -\frac{6t^2 + 32}{(3t^2 + 16)^2} \]

\[ = -\frac{2(3t^2 - 16)}{(3t^2 + 16)^2} \]

\[ C'(1) = -\frac{2(3(1)^2 - 16)}{(3(1)^2 + 16)^2} = -\frac{2(3 - 16)}{(3 + 16)^2} = \frac{-2(-13)}{19^2} \]

\[ = \frac{26}{361} \approx 0.07202 \]

Concentration increasing at a rate of 0.072 units per hour.

\[ \lim_{t \to \infty} \frac{2t}{3t^2 + 16} = \lim_{t \to \infty} \frac{2t}{3t^2 + 16} = \frac{0}{\infty} = 0 \]

In the long run, the concentration goes to zero.
c) Where does concentration start to decrease?
\[
C'(t) = 0 \quad \text{(Slope goes from pos. to neg)}
\]
\[
\frac{-2(3t^2-16)}{(3t^2+16)^2} = 0
\]
\[
\Rightarrow -2(3t^2-16) = 0
\]
\[
3t^2-16 = 0
\]
\[
3t^2 = 16
\]
\[
t^2 = \frac{16}{3}
\]
\[
t = \pm \sqrt{\frac{16}{3}}
\]
\[
t = \pm \frac{4}{\sqrt{3}}
\]
\[
+ \frac{4}{\sqrt{3}} \quad t > 0
\]
\[
\approx +2.3094
\]

The concentration starts declining after about 2.3 hours.
Normal line means a line that is perpendicular to the tangent line at a point.

Recall, slopes of parallel lines:
\[ y_1 = m_1 x + b_1, \quad y_2 = m_2 x + b_2 \]
If \( m_1 = \frac{a}{b} \), then \( m_2 = -\frac{b}{a} \)

Find the normal line to \( f(x) = x^2 + 3x - 5 \) at \((0, -5)\).

\[ f'(x) = 2x + 3 \]
\[ f'(0) = 3 \]
\[ m_1 = 3 \quad \perp \quad m_2 = -\frac{1}{3} \]

\[ y - y_1 = m(x - x_1) \]
\[ y + 5 = -\frac{1}{3} (x - 0) \]
\[ y = -\frac{1}{3} x - 5 \]
Higher order derivatives

Taking the derivative of something that is a rate itself.  
ex. position/velocity/acceleration.

r.o.c of position (mi) is velocity (mph).  
r.o.c of velocity (mph) is acceleration (mph/hr).

\[
\text{Dist.} \quad \text{Traveled 65} \\
\text{Fr} \quad \text{1hr}
\]

\[
\text{position} \Rightarrow f(x) = 65x \\
\text{velocity} \Rightarrow f'(x) = 65 \\
\text{acceleration} \Rightarrow f''(x) = 0
\]

Constant r.o.c 65mph.  
Acceleration? zero if speed is constant.

The second derivative

The second derivative is the derivative of the first derivative of the function \( f(x) \).  
Notation: if \( y = f(x) \),  
\[
\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)
\]

If time...  
\[ f(x) = 84x^5 + \frac{3}{2}x^{-\frac{1}{2}} + \frac{2}{x^3} + 2 \]

\[ f''(x) = 84x^5 + \frac{3}{2\sqrt{x^3}} + \frac{2}{x^3} + 2 \]

\[ f''(x) = 84x^5 + \frac{3}{2\sqrt{x^3}} + \frac{2}{x^3} + 2 \]
**Problem:**
f(x) = (2x^2 + 1)(x - 2)  find f''

\[
f'(x) = (2x^2 + 1)(1) + (x - 2)(4x)
\]
\[
= 2x^2 + 1 + 4x^2 - 8x
\]
\[
= 6x^2 - 8x + 1
\]

\[
f''(x) = 12x - 8
\]

**Problem:**
If height of an object is given by

\[
H(t) = -16t^2 + 32t + 280
\]

Find acceleration. (Find \(H''(t)\))

Velocity,
\[
H'(t) = -32t + 32
\]
\[
H''(t) = -32
\]

The acceleration is negative which means the speed of the object is constantly decreasing. (Without external forces, this makes sense).

**Notation for the n-th derivative (Applications):**
\[
\frac{d^n f}{dx^n} \quad \text{or} \quad f^{(n)}(x)
\]

Find \(f^{(5)}(x)\) if \(f(x) = 3x^6 + 4x^4 + 2x^3 - 5\)

\[
f' = 18x^5 + 16x^3 + 6x
\]
\[
f'' = 90x^4 + 48x^2 + 6
\]
\[
f''' = 360x^3 + 96x
\]
\[
f^{(4)} = 1080x^2 + 96
\]

\[
f^{(5)} = 2160x
\]
Do not include $S_3$ in $2.3$ Hz

$P(x) = \frac{100\sqrt{x}}{0.03x^2 + 9}$

$P'(x) = \frac{(0.03x^2 + 9)(\frac{100}{2\sqrt{x}}) - 100\sqrt{x}(0.06x)}{(0.03x^2 + 9)}$

b) Hint: For what values of $x$ is $P(x)$ increasing? Decreasing

$P'(x) > 0 \quad P(x)$ increasing

$P'(x) = 0 \quad x = 10$

$P'(x) < 0 \quad P(x)$ decreasing

Place where $P$ goes from increasing to decreasing

If $\frac{a}{b} = 0$

$(0.03x^2 + 9)\frac{100}{2}(x^{-\frac{1}{2}}) - 100x^{\frac{1}{2}}(0.06x) = 0$

Divide by $100$

$0.03x^2 + 9 = \frac{1}{2} \cdot x^{\frac{1}{2}} - x^{\frac{1}{2}} \cdot 0.06x = 0$

$x^{\frac{1}{2}} [(0.03x^2 + 9)\frac{1}{2} - x^1 \cdot 0.06x] = 0$

$x^{\frac{1}{2}} = 0$

$0.03x^2 + 9 = 0$

$x = 0$

$x = 0$

$\sqrt{x} = 0$

$0.03x^2 = 9$

$0.03x^2 = 9$

$x = 10$

$0.015$

$\frac{1.0}{.10}$

$0.015$

$0.081\div 0.015$

$0.015$

$0.06$

$\frac{1.0}{.10}$

$0.015$

$0.045$

$6.045 \times 10^3$

$100.045$
The second derivative

The derivative of its derivative.

If \( y = f(x) \), the second deriv. is denoted by \( \frac{d^2y}{dx^2} \) or \( f''(x) \).

**Example:** Find \( f''(x) \) of \( f(x) = 6x^5 + 3x^3 - 2x + 1 \)

\[
\begin{align*}
  f'(x) &= 6(5x^4) + 3(3x^2) - 2 \\
  &= 30x^4 + 9x^2 - 2 \\
  &\text{Simplify completely then take 2nd deriv.}
\end{align*}
\]

\[
\begin{align*}
  f''(x) &= 30(4x^3) + 18(2x) \\
  &= 120x^3 + 36x
\end{align*}
\]

**Example:** Find second deriv. of \( y = x^2(8x+1) \) prod. rule or mult. out 1st.

**Example:** Find all points where the tangent line is horizontal. \( f'(x) = 0 \) solve for \( x \).

\[
\begin{align*}
  f(x) &= x^3 - x^2 - 2x + x^2 - x - 2 \\
  &= x^3 - 3x - 2 \\
  f'(x) &= 3x^2 - 3 = 0 \\
  &= 3x^2 = 3 \\
  x &= \pm 1 \\
\end{align*}
\]

Ordered pairs: \( (1,?) \) \( (-1,?) \)