Section 2.3 Product and Quotient Rules and Higher Order Derivatives

We have sum/difference rules ... but what about product rule.

If $f(x)=x^3$ and $g(x)=x^4$ then
$f'(x)=3x^2$ and $g'(x)=4x^3$, so
so $f'(x)g'(x)=(3x^2)(4x^3)$
$=12x^5$

different

But...
$f(x)\cdot g(x)=(x^3)(x^4)$
$=x^7$
and $[f(x)\cdot g(x)]'=7x^6$

We know this is okay but there's a special product rule.

The Product Rule

$$\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$$

or

$(f\cdot g)'=fg'+gf'$ "1st times deriv. of 2nd, + 2nd times deriv. of 1st"

proof pg 138

Let try again...
$f(x)=x^3$ $g(x)=x^4$

$[f(x)g(x)]'=x^3(4x^3)+x^4(3x^2)$
$=4x^6+3x^6$
$=7x^6$

Typically we have to use the product rule & can't check as we did above.

$f(x) = 2x(x-1)$ easy just mult out & use power rule.

$f(x) = 2x^2 - 2x$

$\boxed{f'(x) = 4x - 2}$

What about uglier ones?

ex) $h(x) = \underbrace{(3x^4 + 6x - 2)}_{f(x)}\underbrace{(5 - x)}_{g(x)}$ mult out? easier to use product rule.

Rule $h'(x) = f \cdot g' + g f'$

$h'(x) = (3x^4 + 6x - 2)(-1) + (5-x)(12x^3 + 6)$

$= -3x^4 - 6x + 2 + 60x^3 + 30 - 12x^4 - 6x$

$\boxed{h'(x) = -15x^4 + 60x^3 - 12x + 32}$

Pieces

$f = 3x^4 + 6x - 2$

$f' = 12x^3 + 6$

$g = 5 - x$

$g' = -1$

Skip ex) $y = (5x^3 + 4\sqrt{x} + x)\left(\frac{1}{x} - \frac{x}{3}\right)$

Take derivative & do not simplify!

$y' = (5x^3 + 4\sqrt{x} + x)\left(\frac{-1}{x^2} - \frac{1}{3}\right) + \left(\frac{1}{x} - \frac{x}{3}\right)\left(15x^2 + \frac{4}{2\sqrt{x}} + 1\right)$

Skip ex. $f(x) = \frac{1}{2}\underbrace{(x^4 - 3x^2 + 1)}_{f}\underbrace{\left(\frac{x}{3} + \frac{2}{x}\right)}_{g}$

$f = x^4 - 3x^2 + 1$

$f' = 4x^3 - 6x$

$g = \frac{1}{3}x + 2x^{-1}$

$g' = \frac{1}{3} - 2x^{-2}$

$= \frac{1}{3} - \frac{2}{x^2}$

$f'(x) = \frac{1}{2}\left[(x^4 - 3x^2 + 1)\left(\frac{1}{3} - \frac{2}{x^2}\right) + \left(\frac{x}{3} + \frac{2}{x}\right)(4x^3 - 6x)\right]$

$= \frac{1}{2}\left[\frac{1}{3}x^4 - 2\frac{x^4}{x^2} - \frac{3x^2}{1}\frac{1}{3} + 6\frac{x^2}{x^2} + \frac{1}{3} - \frac{2}{x^2} + \frac{4x^3}{1} \cdot \frac{x}{3} - \frac{6x}{1}\left(\frac{x}{3}\right) + \frac{2}{x}\frac{4x^{3^2}}{1} - \frac{6x}{1}\frac{2}{x}\right]$

$= \frac{1}{2}\left[\frac{1}{3}x^4 - 2x^2 - x^2 + 6 + \frac{1}{3} - \frac{2}{x^2} + \frac{4}{3}x^4 - 2x^2 + 8x^2 - 12\right]$

$= \frac{1}{2}\left[\frac{1}{3}x^4 + \frac{4}{3}x^4 + 3x^2 - \frac{6}{1}\left(\frac{3}{3}\right) + \frac{1}{3} - \frac{2}{x^2}\right]$

$\boxed{= \frac{1}{2}\left[\frac{5}{3}x^4 + 3x^2 - \frac{17}{3} - \frac{2}{x^2}\right]}$

For the curve

$y = (x-1)(x^2-8x+7)$

a) Find dy/dx

b) Find the equation of the tangent line at $x=2$.

c) Find all points (x,y) where the tangent line is horizontal.

a) $\frac{dy}{dx} = (x-1)(2x-8) + (x^2-8x+7)(1)$

$= 2x^2-8x-2x+8 + x^2-8x+7$

$= 3x^2-18x+15$

$\boxed{dy/dx = 3x^2-18x+15}$

b) $x=2 \Rightarrow f(2) = (2-1)((2)^2-8(2)+7) = 4-16+7 = 11-16 = -5$

$(2,-5)$
$x_1 \quad y_1$

$m = \left.\frac{dy}{dx}\right|_{x=2} = 3(2)^2-18(2)+15$

$= 3(4)-36+15$

$= 12-36+15$

$= 27-36$

$m = -9$

15
$\underline{12}$
27

36
$\underline{-27}$
9

$y+5 = -9(x-2)$

$y+5 = -9x+18$

$y = -9x+18-5$

$\boxed{y = -9x+13}$

c) Horizont. $\Rightarrow m=0$ so set $f'(x)$ equal to zero and solve for x.

$0 = 3x^2-18x+15$

$\frac{0}{3} = \frac{3(x^2-6x+5)}{3}$

$0 = x^2-6x+5$

$0 = (x-5)(x-1) \quad x = 5, 1$

$x=5$

$f(5) = (5-1)(5^2-8(5)+7)$

$= 4(25-40+7)$

$= 4(32-40)$

$= 4(-8) = -32$

$x=1$

$f(1) = (1-1)(\;)$

$= 0$

Horizontal tangents occur at $(5,-32)$ & $(1,0)$

Quotient Rule

common error $\left(\frac{f}{g}\right)' = \frac{f'}{g'}$ Not!

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\frac{d}{dx}[f(x)] - f(x)\,\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

or

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

memorization trick

$$\left(\frac{hi}{lo}\right)' = \frac{lo\,d(hi) - hi\,d(lo)}{lo\,lo}$$

ex. $Q(x) = \frac{2}{x}$

$$Q'(x) = \frac{\cancel{x \cdot 0} - 2(1)}{x^2}$$

$$= \boxed{\frac{-2}{x^2}}$$

Here!

ex. $f(x) = \frac{(2x-1)(x+3)}{x+1}$

mult. numerator 1st or you have to use both the product & quotient rule.

$$f(x) = \frac{2x^2+6x-x-3}{x+1}$$

$$= \frac{2x^2+5x-3}{x+1}$$

$$f'(x) = \frac{(x+1)(4x+5)-(2x^2+5x-3)}{(x+1)^2}$$

$$= \frac{4x^2+\cancel{5x}+4x+5-2x^2-\cancel{5x}+3}{(x+1)^2}$$

$$= \frac{2x^2+4x+8}{(x+1)^2}$$

$$\boxed{= \frac{2(x^2+2x+4)}{(x+1)^2}}$$

ex. $Q(x) = \frac{x^2}{x+5}$

$$Q'(x) = \frac{(x+5)(2x) - x^2(1)}{(x+5)^2}$$

$$= \frac{2x^2+10x-x^2}{(x+5)^2}$$

$$= \frac{x^2+10x}{(x+5)^2}$$

$$= \boxed{\frac{x(x+10)}{(x+5)^2}}$$

Here!

ex. $g(t) = \frac{t^2+\sqrt{t}}{2t+5}$

$$g'(t) = \frac{(2t+5)\left(2t+\frac{1}{2\sqrt{t}}\right) - (t^2+\sqrt{t})(2)}{(2t+5)^2} =$$

Exam leave it here

HW

ex. cont

$$g'(t) = \frac{\underline{4t^2} + 2t\frac{1}{2\sqrt{t}} + 10t + \frac{5}{2\sqrt{t}} - \underline{2t^2} + 2\sqrt{t}}{(2t+5)^2}$$

$$\boxed{= \frac{2t^2 + 10t + \frac{t}{\sqrt{t}} + \frac{5}{2\sqrt{t}} + 2\sqrt{t}}{(2t+5)^2}}$$

clean it up! (Algebra)

ex. Find the equation of the tangent line to the given curve at $x = x_0$

$$y = \frac{x+7}{5-2x} \qquad x_0 = 0$$

$$y' = \frac{(5-2x)(1) - (x+7)(-2)}{(5-2x)^2}$$

$$= \frac{5 - \cancel{2x} + \cancel{2x} + 14}{(5-2x)^2}$$

$$y' = \frac{19}{(5-2x)^2}$$

$x = 0$

$$y'\Big|_{x=0} = \frac{19}{5^2} = \frac{19}{25} = m \qquad y\Big|_{x=0} = \frac{7}{5}$$

$$\left(0, \frac{7}{5}\right)$$

$$y - \frac{7}{5} = \frac{19}{25}(x - 0)$$

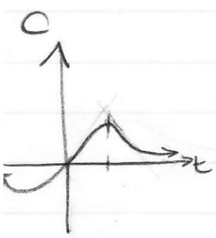
$$\boxed{y = \frac{19}{25}x + \frac{7}{5}}$$

Skip

Applications

Concentration of a pain killer in the bloodstream t hours after ingesting is given by

$$C(t) = \frac{2t}{3t^2+16} \qquad t \text{ in hrs}$$



$t>0$

a) At what rate is the concentration changing after 1 hr?

b) What happens to the concent. in the long-run ($t \to \infty$)

c) When does the concent. start to decrease?

a) $$C'(t) = \frac{(3t^2+16)(2) - (2t)(6t)}{(3t^2+16)^2}$$

$$C'(t) = \frac{6t^2+32-12t^2}{(3t^2+16)^2}$$

$$= \frac{-6t^2+32}{(3t^2+16)^2}$$

$$= \frac{-2(3t^2-16)}{(3t^2+16)^2}$$

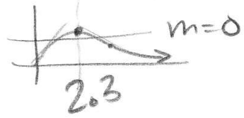
$$C'(1) = \frac{-2(3(1)^2-16)}{(3(1)^2+16)^2} = \frac{-2(3-16)}{(3+16)^2} = \frac{-2(-13)}{19^2}$$

$$= \frac{26}{361} \approx 0.07202$$

Concent. Increasing at a rate of 0.072 units per hour.

b) $$\lim_{t\to\infty} \frac{2t}{3t^2+16} = \lim_{t\to\infty} \frac{\frac{2t}{t^2} \to 0}{\frac{3t^2}{t^2} + \frac{16}{t^2} \to 0} = \frac{0}{3} = \boxed{0}$$

In the long run the concen. goes to zero.



Use T.I. to find max you'd get 2.3 !!

c) When does concentration start to decrease?

$C'(t) = 0$ (slope goes from pos. to neg)

$\frac{a}{b} = 0$ if $a = 0$

$$\frac{-2(3t^2-16)}{(3t^2+16)^2} = 0$$

$$\Rightarrow -2(3t^2-16) = 0$$

$$3t^2 - 16 = 0$$

$$3t^2 = 16$$

$$t^2 = \frac{16}{3}$$

$$t = \pm\sqrt{\frac{16}{3}}$$

$$= \pm\frac{4}{\sqrt{3}} \qquad t > 0$$

$$\approx +2.3094$$

The concentration starts declining after about 2.3 hours.

2.3 - 31, 33, 35, 37, 40, 42, 43, 45, 46, 49, 53

~~Ex.~~ Normal line means a line that is perpendicular to the tangent line at a point.

Recall, slopes of $\perp$ lines

$$y_1 = m_1 x + b_1 \qquad y_2 = m_2 x + b_2$$

$$\text{If } m_1 = \frac{a}{b} \text{ then } m_2 = -\frac{b}{a}$$

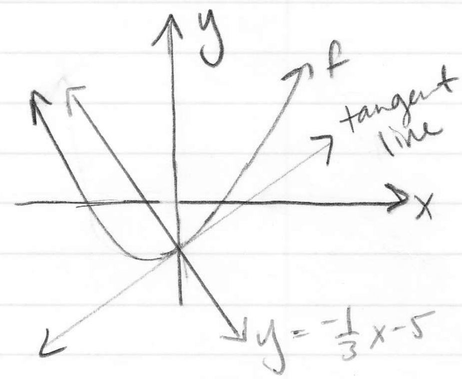
Find the normal line to $f(x) = x^2 + 3x - 5$ at $(0, -5)$

(Find slope of tangent line @ $x = 0$, then use neg. reciprocal)

$f'(x) = 2x + 3$

$f'(0) = 3$

$m_1 = 3 \quad \perp \quad m_2 = \frac{-1}{3}$



$y - y_1 = m(x - x_1)$

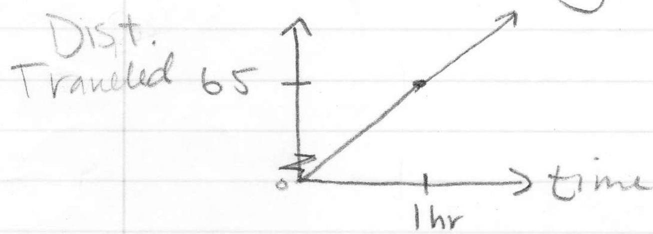
$y + 5 = \frac{-1}{3}(x - 0)$

$\boxed{y = -\frac{1}{3}x - 5}$

Higher order derivatives

☆ Taking the derivative of something that is a rate itself.
ex. position/velocity/acceleration

r.o.c of position (mi) is velocity (mph)
r.o.c of velocity (mph) is acceleration (mph/hr)



position $\Rightarrow f(x) = 65x$
velocity $\Rightarrow f'(x) = 65$
acceleration $\Rightarrow f''(x) = 0$

Constant r.o.c 65mph.
Acceleration? zero if speed is constant.

The second derivative

The second derivative is the derivative of the first derivative of the function $f(x)$.
Notation: If $y = f(x)$,

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)$$

☆ next pg

If time...

ex. Find the second derivative of

$$f(x) = 2x^7 - 6\sqrt{x} + \frac{1}{x} + 2$$
$$= 2x^7 - 6x^{1/2} + x^{-1} + 2$$
$$f'(x) = 14x^6 - \frac{6}{2}x^{-1/2} - 1x^{-2} + 2$$
$$f''(x) = 84x^5 + 3\left(\tfrac{1}{2}\right)x^{-\frac{1}{2}-\frac{2}{2}} + 2x^{-3} + 2$$

$$\boxed{f''(x) = 84x^5 + \frac{3}{2\sqrt{x^3}} + \frac{2}{x^3} + 2}$$

$14 \times 6 = 84$

☆ ex. $f(x) = (2x^2+1)(x-2)$ find f''

$f'(x) = (2x^2+1)(1) + (x-2)(4x)$
$= 2x^2+1+4x^2-8x$
$= 6x^2-8x+1$

$\boxed{f''(x) = 12x - 8}$

If time...

Applic

If height of an object is given by

$p' = v$
$v' = a$

$H(t) = -16t^2 + 32t + 280$

Find acceleration. (find $H''(t)$)

Velocity
$H'(t) = -32t + 32$
$H''(t) = -32$

The acceration is negative which means the speed of the object is constantly decreasing. (w/out external forces, this makes sense)

Notation for the n^{th} derivative (Applications)

$\frac{d^nf}{dx^n}$ or $f^{(n)}(x)$ with parentheses

Find $f^{(5)}(x)$ if $f(x) = 3x^6 + 4x^4 + 2x^3 - 5$

$18 \times 5 = 90$ $\quad 16 \times 3 = 48$ $\quad 90 \times 4 = 360$ $\quad 48 + 48 = 96$ $\quad 360 \times 3 = 1080$

$f' = 18x^5 + 16x^3 + 6x$
$f'' = 90x^4 + 48x^2 + 6$
$f''' = 360x^3 + 96x$
$f^{(4)} = 1080x^2 + 96$

$\boxed{f^{(5)} = 2160x}$

Do not include 53 in 2.3 HW

Do 52 instead

UGH

$$P(x) = \frac{100\sqrt{x}}{0.03x^2+9}$$

2.3 #53

$$P'(x) = \frac{(0.03x^2+9)\left(\frac{100}{2\sqrt{x}}\right) - 100\sqrt{x}\,(0.06x)}{(0.03x^2+9)}$$

b) Hint: For what values of x is P(x) increasing? Decreasing
$0<x<10$ $x>10$

$P'>0$
$P'=0$
$P'<0$

P(x) increasing $\Rightarrow P'(x) > 0$

P(x) decreasing $\Rightarrow P'(x) < 0$

place where P goes from increasing to decreasing

If $\frac{a}{b} = 0$

$\Rightarrow a = 0$

$b \neq 0$.

$$(0.03x^2+9)\frac{100}{2}(x^{-1/2}) - 100x^{1/2}(0.06x) = 0$$

Divide by 100

$$(0.03x^2+9)\cdot \frac{1}{2}\cdot x^{-1/2} - x^{1/2}\cdot 0.06x = 0$$

factor out $x^{1/2}$

$$x^{-1/2}\left[(0.03x^2+9)\tfrac{1}{2} - x^1\cdot 0.06x\right] = 0$$

$x^{-1/2} = 0$

$\sqrt{x} = 0$

$\boxed{x = 0}$

$$\frac{0.03x^2}{2} + \frac{9}{2} - 0.06x^2 = 0$$

$0.015x^2 - 0.06x^2 = -4.5$

$-0.045x^2 = -4.5$

$x^2 = 100$

$\boxed{x = 10}$

$$2\overline{)0.030}$$ = 0.015
2
10
-10
0

0.015
-.06
5

0.060
0.015
045

100

0.045)4.500
45

The second derivative

The derivative of its derivative.

If $y=f(x)$, the second deriv. is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$

~~#10. 2 of the~~

ex. Find $f''(x)$ of $f(x)=6x^5+3x^3-2x+1$

$$f'(x)=6(5x^4)+3(3x^2)-2$$
$$=30x^4+9x^2-2$$

simplify completely then take 2nd deriv.

$$f''(x)=30(4x^3)+9(2x)$$
$$\boxed{=120x^3+18x}$$

---

ex. find second deriv. of

$$y=x^2(3x+1)$$

prod. rule or mult. out 1st.

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ex Find all points where the tangent line is horizontal. $f'(x)=0$ solve for x.

$$f(x)=x^3-\cancel{x^2}-2x\cancel{+x^2}-x-2$$
$$=x^3-3x-2$$
$$f'(x)=3x^2-3=0$$
$$3x^2=3$$

$x^2=1$

$x=\pm 1$

ordered pairs $(1, \ )$ $(-1, \ )$

y y