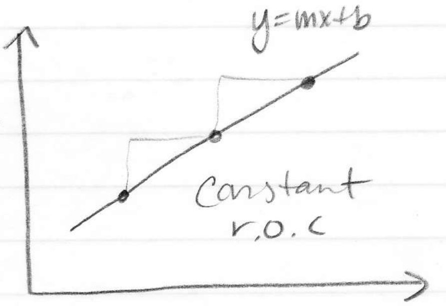
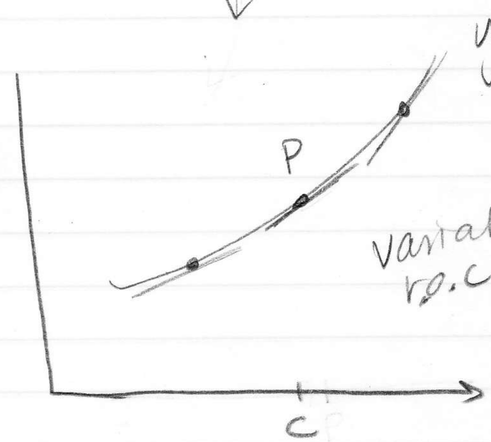
Section 2.1 The Derivative (Two classes to cover)
Calculus is the mathematics/study of change. Primary tool to study change: Differentiation

Slopes and Rates of Change

Linear Functions ($L(x) = mx+b$) have a constant rate of change m with respect to x.
r.o.c

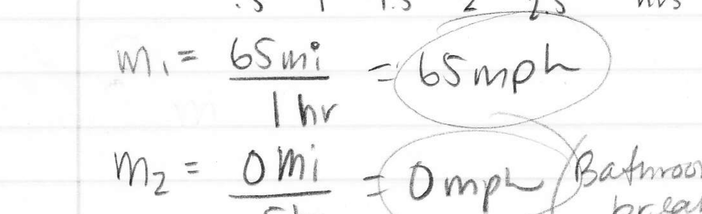
Non-linear Functions - rate of change is not constant, but varies with x.





R.O.C is given by the slope of the tangent (touches @ one pt) line at $P(c, f(c))$.
i.e. the slope at that point.

Distance vs. Velocity



$m_1 = \frac{65\text{mi}}{1\text{hr}} = 65\text{mph}$

$m_3 = \frac{40\text{mi}}{1\text{hr}} = 40\text{mph}$

$m_2 = \frac{0\text{mi}}{.5\text{hr}} = 0\text{mph}$ (Bathroom break)

Slope = Rate of Change

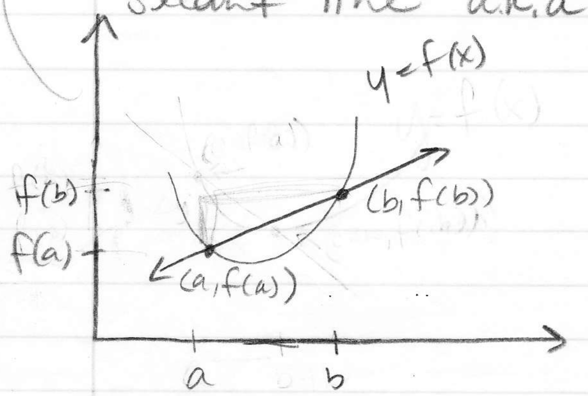
Average velocity/rate of change

Graphically

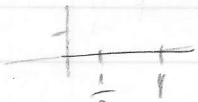
$$\text{Ave R.O.C.} = \frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(b) - f(a)}{b-a}$$

(touches @ two points)

Secant line a.k.a average rate of change



ex. Let $f(x) = 3x^2 - x$
Find the average r.o.c with respect to x as x changes from $x = \frac{1}{2}$ to $x = 1$



Pieces

$f(a)$
$= f(\frac{1}{2})$
$= 3(\frac{1}{2})^2 - \frac{1}{2}$
$= 3(\frac{1}{4}) - \frac{1}{2}$
$= \frac{3}{4} - \frac{1}{2}(\frac{2}{2})$
$= \frac{3}{4} - \frac{2}{4}$
$= \frac{1}{4}$

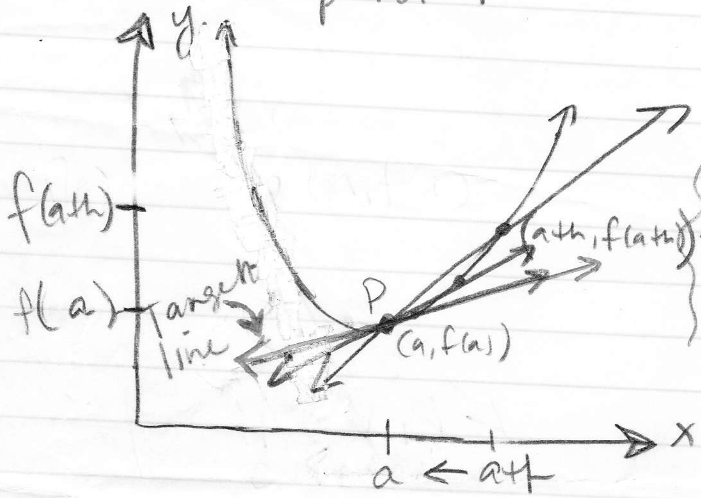
$f(b)$
$= f(1)$
$= 3(1)^2 - 1$
$= 3 - 1$
$= 2$

$$\text{Rate}_{AVE} = \frac{2 - \frac{1}{4}}{\frac{1}{2}} = \frac{\frac{8}{4} - \frac{1}{4}}{\frac{1}{2}}$$
$$= \frac{7}{4} \div \frac{1}{2}$$
$$= \frac{7}{\cancel{4}_2} \cdot \cancel{2} = \boxed{\frac{7}{2}}$$

Secant lines vs. tangent lines

- Secant line touches at two points 
- Tanget line touches only at one point 

To approx. slope of tangent line we use secant lines thru point P



As $h \to a$, secant line → tangent line at P.

$$Ave_{r.o.c} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

Instantaneous velocity / r.o.c of f(x) at x=a
instant in time

$$\text{Inst. r.o.c} = \lim_{h\to 0} \cancel{\text{Rate}_{ave}} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

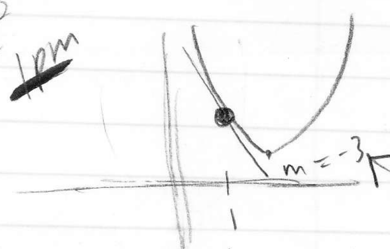
= Slope of tangent line at P.

ex) Compute the instantaneous r.o.c at $x=1$ ← a
if $f(x) = x^2 - 5x$

Pieces to plug in.

$$f(a+h) = f(1+h)$$
$$= (1+h)^2 - 5(1+h)$$
$$= h^2 + 2h + 1 - 5 - 5h$$
$$= h^2 - 3h - 4$$

$$f(a) = f(1)$$
$$= (1)^2 - 5(1)$$
$$= -4$$



The inst. r.o.c at $x=1$ is -3. (slope of tang. is -3)

day 1

$$\text{Inst. r.o.c} = \lim_{h\to 0} \frac{h^2-3h-4-(-4)}{h} = \lim_{h\to 0} \frac{h^2-3h}{h} = \lim_{h \to 0} \frac{h(h-3)}{h} = \lim_{h\to 0} h-3 = \boxed{-3}$$

What if we were asked to find ...

ex. Find the inst. r.o.c when $x=9$
if $f(x)=\sqrt{x}$

$f(a+h) = f(9+h)$
$= \sqrt{9+h}$

$f(a) = f(9) = 3$

$$\lim_{h\to 0} \frac{(\sqrt{9+h}-3)}{h}\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right)$$

$$= \lim_{h\to 0} \frac{(9+h)-9}{h(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3}$$

$$= \frac{1}{\sqrt{9}+3}$$

$$= \boxed{\frac{1}{6}}$$

The inst. r.o.c at $x=9$ is $\frac{1}{6}$.

Imagine if you had to do this for <u>many</u> x values → Formula in general would be nice! → the derivative

The derivative

The derivative of the function $f(x)$ w.r.t x is the function $f'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

"f prime of x"

difference quotient from ch. 1

Vocab: To take derivative: <u>differentiate</u>
process: <u>differentiation</u>

$f(x)$ <u>differentiable</u> at $x=c$ if $f'(c)$ exists

Notations

$\frac{dy}{dx}$, $f'(x)$, $\frac{df}{dx}$, $D_x[f]$, $\dot{f}(x)$, $\frac{d}{dx}[f]$

"The derivative of f w.r.t x"

$f'(c) = \left.\frac{df}{dx}\right|_{x=c}$

11am

Back to example: Compute derivative

ex. $f(x) = \sqrt{x}$ find instant. r.o.c at $x = \frac{1}{4}, x = 1, x = 0$ using the defn of derivative. $x = 9$.

Leave variable x in now

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{x+h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

No short cut rules until 2.2!

Note: 0 is in the domain of $f(x)$, but not $f'(x)$.
→ Functions & their deriv. do not always have same domain.

Now find

② $f'(1) = \frac{1}{2\sqrt{1}}$ $\boxed{= \frac{1}{2}}$

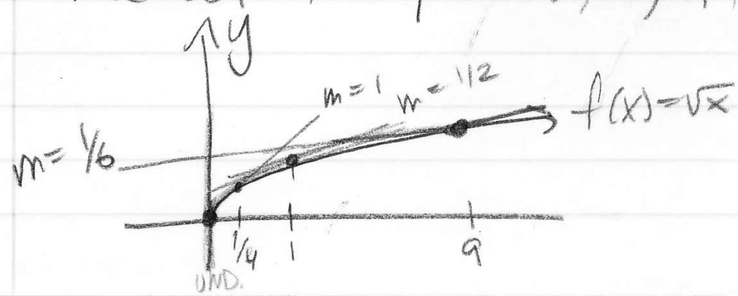
① $f'(\frac{1}{4}) = \frac{1}{2\sqrt{\frac{1}{4}}} = \frac{1}{2(\frac{1}{2})} = \frac{1}{1}$ $\boxed{= 1}$

You can find f' at any point easily.

③ $f'(9) = \frac{1}{2\sqrt{9}}$ $\boxed{= \frac{1}{6}}$

④ $f'(0) = $ UND

Once again, all of these are the slopes of the tangent line at the points, 1, $\frac{1}{4}$ & 9



$f'(c)$, the derivative of $f(x)$ at the point $x = c$ is equivalent to

→ The <u>slope of the tangent</u> line to $y = f(x)$ at $(c, f(c))$

$$m_{tan} = f'(c).$$

→ The <u>instantaneous r.o.c</u> of $f(x)$ w.r.t x when $x = c$ and is $f'(c)$.

☆ pg 1

ex)
a) Compute the derivative &
b) find the <u>slope</u> of the tangent line of the function at the given point.
c) equation

$f(x) = 2x^2+4 \qquad x = 3$

$$m_{tan} = f'(x) = \lim_{h\to 0} \frac{\overbrace{2(x+h)^2+4}^{f(x+h)} - \overbrace{(2x^2+4)}^{f(x)}}{h}$$

Should go away!! :)

$$= \lim_{h\to 0} \frac{2(x^2+2xh+h^2)+4-2x^2-4}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2}+4xh+2h^2+\cancel{4}-\cancel{2x^2}-\cancel{4}}{h}$$

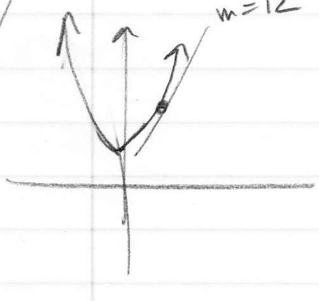
$$= \lim_{h\to 0} \frac{4xh+2h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4x+2h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 4x+2h$$

$$= 4x+2(0)$$

$$= 4x \qquad \boxed{f'(x) = 4x}$$



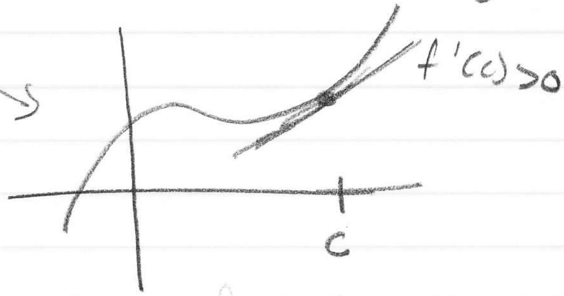
$f'(3) = 4(3)$
$\boxed{=12}$ ← slope of tangent line at $x=3$

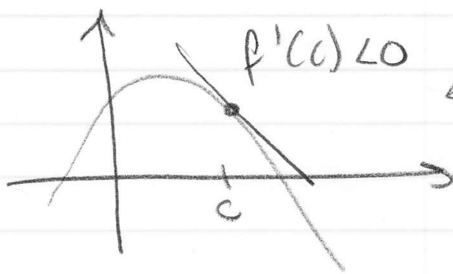
Find the equation of the tangent line.

$m = 12$ point? $(3, f(3))$ $\quad f(3) = 2(3)^2+4$
$(3, 22)$ $\quad = 18+4$
$\quad = 22$

★ Sign of $f'(x)$

f is <u>increasing</u> at $x=c$ if $f'(c)>0$
f is <u>decreasing</u> at $x=c$ if $f'(c)<0$



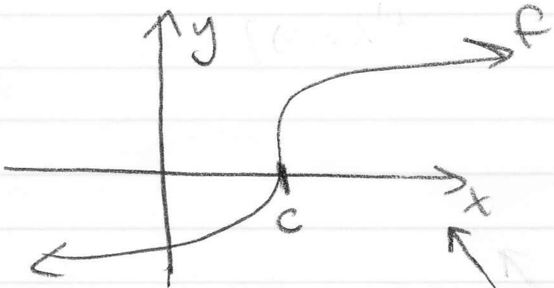


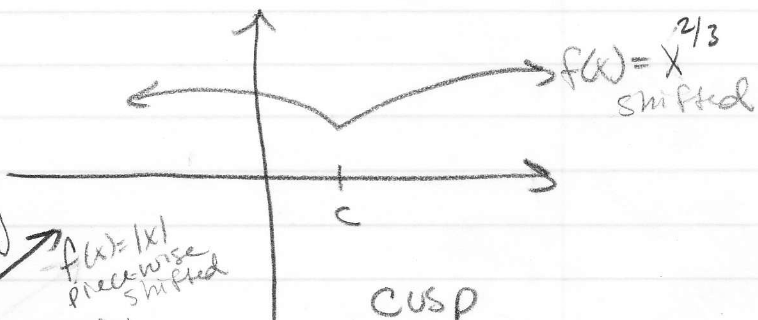
If a function is differentiable at $x=c$, then it is also continuous at $x=c$.

Diff $\Longrightarrow$ Continuous (not true other way)

Functions are <u>not</u> differentiable when there are abrupt changes in the graph.

$f(x)=x^{1/3}$ shifted





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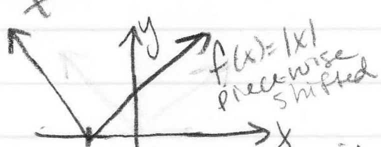

Vertical tangent   sharp point

Both <u>continuous</u>, but <u>not</u> differentiable at $x=c$.

Polynomials are differentiable at $\mathbb{R}$
Rationals " " on their domains.

<u>extra</u>

Let $f(x)=x^3$

a) $m_{sec}=?$ $x=1$ & $x=1.1$

b) $m_{tan}=?$ $x=1$

compare

ex. con't $m=12$ $(3, 22)$ x_1 y_1

$y - y_1 = m(x - x_1)$

equivalent!

$y - f(c) = f'(c)(x-c)$

$y - 22 = 12(x-3)$

$y - 22 = 12x - 36$

$y = 12x - 36 + 22$

$y = 12x - 14$

$$\begin{array}{r} 12 \\ 12 \\ \underline{12} \\ 36 \\ \underline{-22} \\ 14 \end{array}$$

The equation of the tangent line of $f(x) = 2x^2 + 4$ at the point $x=3$ is $y = 12x - 14$.

If time

ex) compute derivative

$g(t) = \frac{5}{t}$

$$g'(t) = \lim_{h \to 0} \left[\frac{\frac{5}{t+h} - \frac{5}{t}}{h} \right]$$

Get common denom & subtract the two

$$= \lim_{h \to 0} \left[\frac{\frac{5t - 5(t+h)}{(t+h)t}}{h} \right]$$

$$= \lim_{h \to 0} \frac{\frac{\cancel{5t} - \cancel{5t} - 5h}{(t+h)t}}{h}$$

$$= \lim_{h \to 0} \frac{-5\cancel{h}}{t(t+h)} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{-5}{t(t+h)}$$

$$= \frac{-5}{t(t+0)} = \frac{-5}{t^2}$$

$$\boxed{g'(t) = \frac{-5}{t^2}}$$